

A SCHAUDER APPROACH TO DEGENERATE-PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS WITH UNBOUNDED COEFFICIENTS

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ABSTRACT. Motivated by applications to probability and mathematical finance, we consider a parabolic partial differential equation on a half-space whose coefficients are suitably Hölder continuous and allowed to grow linearly in the spatial variable and which become degenerate along the boundary of the half-space. We establish existence and uniqueness of solutions in weighted Hölder spaces which incorporate both the degeneracy at the boundary and the unboundedness of the coefficients. In our companion article [13], we apply the main result of this article to show that the martingale problem associated with a degenerate-elliptic partial differential operator is well-posed in the sense of Stroock and Varadhan.

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1. INTRODUCTION

Motivated by applications to probability theory and mathematical finance [2, 6, 20, 32], we use a Schauder approach to prove existence, uniqueness, and regularity of solutions to a *degenerate*-parabolic partial differential equation with *unbounded*, locally Hölder-continuous coefficients, (a, b, c) with $a = (a_{ij})$ and $b = (b_i)$, generalizing both the *Heston equation* [21] and the *linearization of the porous medium equation* [3, 4, 22],

$$\begin{cases} Lu = f & \text{on } \mathbb{H}_T, \\ u(0, \cdot) = g & \text{on } \overline{\mathbb{H}}, \end{cases} \quad (1.1)$$

where $\mathbb{H} := \mathbb{R}^{d-1} \times (0, \infty)$ (with $d \geq 2$) denotes the half-space $\{x_d \geq 0\}$, and $\mathbb{H}_T := (0, T) \times \mathbb{H}$ is the open half-cylinder with $0 < T < \infty$, and

$$-Lu = -u_t + \sum_{i,j=1}^d x_d a_{ij} u_{x_i x_j} + \sum_{i=1}^d b_i u_{x_i} + cu, \quad \forall u \in C^{1,2}(\mathbb{H}_T). \quad (1.2)$$

The operator L becomes *degenerate* along the boundary $\partial\mathbb{H} = \{x_d = 0\}$ of the half-space but in addition, unlike the linearization of the porous medium equation considered in [3, 4, 22], the coefficients of (1.2) are also permitted to grow linearly with x as $x \rightarrow \infty$ and, even when the coefficients b_i are constant, we do not require that $b_i = 0$ when $i = 1, \dots, d-1$.

In a companion article¹ [13], we apply the main result of the present article (Theorem 1.1) to prove that the martingale problem associated with the degenerate-elliptic partial differential operator acting on $v \in C^2(\overline{\mathbb{H}})$,

$$\mathcal{A}_t v(x) := \frac{1}{2} \sum_{i,j=1}^d x_d a_{ij}(t, x) v_{x_i x_j}(x) + \sum_{i=1}^d b_i(t, x) v_{x_i}(x), \quad (t, x) \in [0, \infty) \times \mathbb{H}, \quad (1.3)$$

is well-posed in the sense of Stroock and Varadhan [36]. In [13], we then prove existence, uniqueness, and the strong Markov property for weak solutions to the associated stochastic differential equation with degenerate diffusion coefficient and unbounded diffusion and drift coefficients with suitable Hölder continuity properties. Finally, in [13], given an Itô process with degenerate diffusion coefficient and unbounded but appropriately regular diffusion and drift coefficients, we prove existence of a strong Markov process, unique in the sense of probability law, whose one-dimensional marginal probability distributions match those of the given Itô process.

1.1. Summary of main results. We describe our results outlined in the preamble to §1. We shall seek a solution, u , to (1.1) in a certain weighted Hölder space $\mathcal{C}_p^{2+\alpha}(\overline{\mathbb{H}}_T)$, given a source function, f , in a weighted Hölder space $\mathcal{C}_p^\alpha(\overline{\mathbb{H}}_T)$ and initial data, g , in a weighted Hölder space $\mathcal{C}_p^{2+\alpha}(\overline{\mathbb{H}})$. These weighted Hölder spaces generalize both the standard Hölder spaces as defined, for example, in [24, 28] and the Hölder spaces defined with the cycloidal metric and introduced, independently, by Daskalopoulos and Hamilton [3] and Koch [22]. We defer a detailed description of these Hölder spaces to §2.1. However, the essential features of our Hölder spaces are that (i) near the boundary, $x_d = 0$, of the half-space cylinder \mathbb{H}_T , our Hölder spaces are equivalent to those of Daskalopoulos, Hamilton, and Koch and account for the degeneracy of the operator L , (ii) polynomial weights in the definition of our Hölder spaces allow for coefficients $(x_d a, b, c)$ in (1.2) with up to linear growth near $x = \infty$ in the half-space cylinder \mathbb{H}_T , and (iii) on compact subsets of the half-space cylinder \mathbb{H}_T , our Hölder spaces are equivalent to standard Hölder spaces.

¹Our longer previous manuscript [11] combined [13] with the present article.

We defer a detailed description of the conditions on the coefficients (a, b, c) defining L in (1.2) to §2.2 — see Assumption 2.2 on the properties of the coefficients of the parabolic differential operator. However, the essential features of the conditions on (a, b, c) in Assumption 2.2 are that (i) the matrix $a = (a_{ij})$ is *uniformly elliptic*, so the *degeneracy* in (1.2) is captured by the common factor x_d appearing in the $u_{x_i x_j}$ terms, (ii) the coefficients $(x_d a, b, c)$ have at most *linear growth* with respect to $x \in \mathbb{H}$ as $x \rightarrow \infty$, (iii) the coefficients (a, b, c) are *locally Hölder continuous* on $\overline{\mathbb{H}}_T$ with exponent $\alpha \in (0, 1)$, (iv) the coefficient c is *bounded above* on \mathbb{H}_T by a constant, and (v) the coefficient b_d is *positive* when $x_d = 0$. We can now state our first main result.

Theorem 1.1 (Existence and uniqueness of solutions to a degenerate-parabolic partial differential equation with unbounded coefficients). *Assume that the coefficients (a, b, c) in (1.2) obey the conditions in Assumption 2.2. Then there is a positive constant p , depending only on the Hölder exponent $\alpha \in (0, 1)$, such that for any $T > 0$, $f \in \mathcal{C}_p^\alpha(\overline{\mathbb{H}}_T)$ and $g \in \mathcal{C}_p^{2+\alpha}(\overline{\mathbb{H}})$, there exists a unique solution $u \in \mathcal{C}^{2+\alpha}(\overline{\mathbb{H}}_T)$ to (1.1). Moreover, u satisfies the a priori estimate*

$$\|u\|_{\mathcal{C}^{2+\alpha}(\overline{\mathbb{H}}_T)} \leq C \left(\|f\|_{\mathcal{C}_p^\alpha(\overline{\mathbb{H}}_T)} + \|g\|_{\mathcal{C}_p^{2+\alpha}(\overline{\mathbb{H}})} \right), \quad (1.4)$$

where C is a positive constant, depending only on $K, \nu, \delta, d, \alpha$ and T .

One of the difficulties in establishing Theorem 1.1 is that the coefficient, $x_d a(t, x)$, becomes degenerate when $x_d = 0$ and is allowed to have linear growth in x , instead of being uniformly elliptic and bounded as in [25, Hypothesis 2.1]. To address the degeneracy of $x_d a(t, x)$ as $x_d \downarrow 0$, we build on the results on [3, Theorem I.1.1] by employing a localization procedure. To address the linear growth of the coefficients $(x_d a, b, c)$ of the parabolic operator L in (1.2), we augment previous definitions of weighted Hölder spaces [3, 22], by introducing a weight $(1 + |x|)^p$, where p is a positive constant depending only on the Hölder exponent $\alpha \in (0, 1)$. The proof of existence does not follow by standard methods, for example, the method of continuity, because $L : \mathcal{C}^{2+\alpha}(\overline{\mathbb{H}}_T) \rightarrow \mathcal{C}_p^\alpha(\overline{\mathbb{H}}_T)$ is not a well-defined operator. In general, the domain of definition of L is a subspace of $\mathcal{C}^{2+\alpha}(\overline{\mathbb{H}}_T)$ which depends on the nature of the coefficients of L , a feature which is not encountered in the case of parabolic operators with bounded coefficients. To circumvent this difficulty, we first consider the case of similar degenerate operators with *bounded* coefficients and then use an approximation procedure to obtain our solution. To obtain convergence of sequences to a solution of our parabolic differential equation (1.1), we prove a priori estimates in the weighted Hölder spaces \mathcal{C}_p^α and $\mathcal{C}_p^{2+\alpha}$.

The conditions in Assumption 2.2 on the coefficients (a, b, c) in (1.2) are mild enough that they allow for many examples of interest in mathematical finance.

Example 1.2 (Parabolic Heston partial differential equation). The conditions in Assumption 2.2 are obeyed by the coefficients of the parabolic Heston partial differential operator,

$$-Lu = -u_t + \frac{y}{2} (u_{xx} + 2\rho\sigma u_{xy} + \sigma^2 u_{yy}) + (r - q - y/2)u_x + \kappa(\theta - y)u_y - ru, \quad (1.5)$$

where $q \geq 0, r \geq 0, \kappa > 0, \theta > 0, \sigma > 0$, and $\rho \in (-1, 1)$ are constants.

Naturally, the conditions in Assumption 2.2 on the coefficients (a, b, c) in (1.2) also allow for the linearization of the generalized porous medium equation.

Example 1.3 (Linearization of the porous medium equation). In their landmark article, Daskalopoulos and Hamilton [3] proved existence and uniqueness of C^∞ solutions, u , to the Cauchy problem

for the porous medium equation [3, p. 899] (when $d = 2$),

$$-u_t + \sum_{i=1}^d (u^m)_{x_i x_i} = 0 \quad \text{on } (0, T) \times \mathbb{R}^d, \quad u(\cdot, 0) = g \quad \text{on } \mathbb{R}^d, \quad (1.6)$$

where $m > 1$ and $g \in L^1(\mathbb{R}^d)$ with $g \geq 0$ compactly supported on \mathbb{R}^d , together with C^∞ -regularity of its free boundary, $\partial\{u > 0\}$. Their analysis is based on an extensive development of existence, uniqueness, and regularity results for the linearization of the porous medium equation near the free boundary and, in particular, their *model linear degenerate operator* [3, p. 901] (generalized from $d = 2$ in their article),

$$-Lu = -u_t + x_d \sum_{i=1}^d u_{x_i x_i} + \nu u_{x_d}, \quad (1.7)$$

where ν is a positive constant. The same model linear degenerate operator (for $d \geq 2$), was studied independently by Koch [22, Equation (4.43)] and, in a remarkable Habilitation thesis, he obtained existence, uniqueness, and regularity results for solutions to (1.6) which complement those of Daskalopoulos and Hamilton [3]. Even when the coefficients in (1.2) are constant, our operator *cannot* be transformed by simple coordinate changes to one of the form (1.7), but rather one of the form (A.7). Similarly, the operator (1.5) *cannot* be transformed by simple coordinate changes to one of the form (1.7), even when the factor y in the coefficients of u_x and u_y in (1.5) is (artificially) replaced by zero.

1.2. Connections with previous research on degenerate partial differential equations.

We provide a brief survey of some related research by other authors on Schauder a priori estimates and regularity theory for solutions to degenerate-elliptic and degenerate-parabolic partial differential equations most closely related to the results described in our article.

The principal feature which distinguishes the Cauchy problem (1.1), when the operator L is given by (1.2), from the linear, second-order, strictly parabolic operators in [26, 28, 24] and their initial-boundary value problems, is the degeneracy of L due to the factor, x_d , in the coefficients of $u_{x_i x_j}$ and, because the coefficient b_d of u_{x_d} in (1.2) is positive, the fact that boundary conditions may be omitted along $x_d = 0$ when we seek solutions, u , with sufficient regularity up to $x_d = 0$.

The literature on degenerate elliptic and parabolic partial differential equations is vast, with the well-known articles of Fabes, Kenig, and Serapioni [8, 9], Fichera [15, 16], Kohn and Nirenberg [23], Murthy and Stampacchia [29, 30] and the monographs of Levendorskiĭ [27] and Oleĭnik and Radkevič [31, 33, 34], being merely the tip of the iceberg.

As far as the authors can tell, however, there has been relatively little prior work on a priori Schauder estimates and higher-order Hölder regularity of solutions up to the portion of the domain boundary where the operator becomes degenerate. In this context, the work of Daskalopoulos, Hamilton, and Rhee [3, 4, 35] and of Koch stands out in recent years because of their introduction of the cycloidal metric on the upper-half space, weighted Hölder norms, and weighted Sobolev norms which provide the key ingredients required to unlock the existence, uniqueness, and higher-order regularity theory for solutions to the porous medium equation (1.6) and the degenerate-parabolic model equation (1.7) on the upper half-space given by the linearization of the porous medium equation in suitable coordinates.

While the Daskalopoulos-Hamilton Schauder theory for degenerate-parabolic operators has been adopted so far by relatively few other researchers, it has also been employed by De Simone, Giacomelli, Knüpfer, and Otto in [5, 18, 17] and by Epstein and Mazzeo in [7].

1.3. Extensions and future work. Motivated by the results obtained in our related “degenerate-elliptic” article [14], it is natural to consider higher-order, interior and boundary regularity and a priori Schauder estimates for solutions $u \in C_s^{k,2+\alpha}(\underline{Q}) \cap C(\bar{Q})$ to an initial-boundary value problem generalization,

$$\begin{cases} Lu = f & \text{on } Q, \\ u = g & \text{on } \bar{\partial}_1 Q, \end{cases} \quad (1.8)$$

of the Cauchy problem (1.1) for the operator L in (1.2). Here, the cylinder $(0, T) \times \mathbb{H}$ has been replaced by a subdomain $Q \subset (0, T) \times \mathbb{H}$ with non-empty “degenerate boundary” portion $\bar{\partial}_0 Q := \text{int}(((0, T) \times \partial\mathbb{H}) \cap \partial Q)$ of the parabolic boundary,

$$\bar{\partial} Q := ((\{0\} \times \mathbb{H}) \cap Q) \cup (((0, T) \times \mathbb{H}) \cap \partial Q),$$

and $\bar{\partial}_1 Q := \bar{\partial} Q \setminus \bar{\partial}_0 Q$ denotes the “non-degenerate boundary” portion of the parabolic boundary, while $\underline{Q} := Q \cup \bar{\partial}_0 Q$.

For reasons we summarize in [12, §1.3], the development of *global* Schauder a priori estimates, regularity, and existence theory for solutions $u \in C_s^{k,2+\alpha}(\bar{Q})$ to (1.8) appears very difficult when the intersection, $\bar{\partial}_0 Q \cap \bar{\partial}_1 Q$, of the “degenerate and non-degenerate boundary” portions is non-empty. In fact, even the development of an existence theory for solutions u to (1.8) just belonging to $C_s^{2+\alpha}(\underline{Q}) \cap C(\bar{Q})$ is already a challenging problem which is not addressed in [3, 4].

While our a priori Schauder estimates rely on the specific form of the degeneracy factor, x_d , of the operator L in (1.2) on a subdomain of the half-space, we obtained weak and strong maximum principles for a much broader class of degenerate-elliptic operators in [10]. Thus, for degenerate-parabolic operators such as

$$-Lv = -v_t + \vartheta \sum_{i,j=1}^d a_{ij} v_{x_i x_j} + \sum_{i=1}^d b_i v_{x_i} + cv \quad \text{on } Q, \quad v \in C^\infty(Q),$$

where $\vartheta \in C_{\text{loc}}^\alpha(\bar{Q})$ and $\vartheta > 0$ on a subdomain $Q \subset (0, T) \times \mathbb{R}^d$ with non-empty “degenerate boundary” portion $\bar{\partial}_0 Q := \text{int}(\{(t, x) \in \partial Q : \vartheta(x) = 0\})$ of the parabolic boundary,

$$\bar{\partial} Q := ((\{0\} \times \mathbb{R}^d) \cap Q) \cup (((0, T) \times \mathbb{R}^d) \cap \partial Q),$$

we plan to develop a priori Schauder estimates, regularity, and existence theory in a subsequent article.

1.4. Outline of the article. In §2, we define the Hölder spaces required to prove Theorem 1.1 (existence and uniqueness of solutions to a degenerate-parabolic partial differential equation on a half-space with unbounded coefficients) and provide a detailed description of the conditions required of the coefficients (a, b, c) in the statement of Theorem 1.1, which we then proceed to prove in §3. Appendices A and B contain proofs for several results which are slightly more technical than those in the body of the article.

1.5. Notation and conventions. We adopt the convention that a condition labeled as an *Assumption* is considered to be universal and in effect throughout this article and so not referenced explicitly in theorem and similar statements; a condition labeled as a *Hypothesis* is only considered to be in effect when explicitly referenced.

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2. WEIGHTED HÖLDER SPACES AND COEFFICIENTS OF THE DIFFERENTIAL OPERATORS

In §2.1, we introduce the Hölder spaces required for the statement and proof of Theorem 1.1, while in §2.2, we describe the regularity and growth conditions required of the coefficients (a, b, c) in Theorem 1.1.

2.1. Weighted Hölder spaces. For $a > 0$, we denote

$$\mathbb{H}_{a,T} := (0, T) \times \mathbb{R}^{d-1} \times (0, a),$$

and, when $T = \infty$, we denote $\mathbb{H}_\infty = (0, \infty) \times \mathbb{H}$ and $\mathbb{H}_{a,\infty} = (0, \infty) \times \mathbb{R}^{d-1} \times (0, a)$. We denote the usual closures these half-spaces and cylinders by $\overline{\mathbb{H}} := \mathbb{R}^{d-1} \times [0, \infty)$, $\overline{\mathbb{H}}_T := [0, T] \times \overline{\mathbb{H}}$, while $\overline{\mathbb{H}}_{a,T} := [0, T] \times \mathbb{R}^{d-1} \times [0, a]$. We write points in \mathbb{H} as $x := (x', x_d)$, where $x' := (x_1, x_2, \dots, x_{d-1}) \in \mathbb{R}^{d-1}$. For $x^0 \in \overline{\mathbb{H}}$ and $R > 0$, we let

$$\begin{aligned} B_R(x^0) &:= \{x \in \mathbb{H} : |x - x^0| < R\}, \\ Q_{R,T}(x^0) &:= (0, T) \times B_R(x^0), \end{aligned}$$

and denote their usual closures by $\bar{B}_R(x^0) := \{x \in \mathbb{H} : |x - x^0| \leq R\}$ and $\bar{Q}_{R,T}(x^0) := [0, T] \times \bar{B}_R(x^0)$, respectively. We write B_R or $Q_{R,T}$ when the center, x^0 , is clear from the context or unimportant.

A parabolic partial differential equation with a degeneracy similar to that considered in this article arises in the study of the porous medium equation [3, 4, 22]. The existence, uniqueness, and regularity theory for such equations is facilitated by the use of Hölder spaces defined by the *cycloidal metric* on \mathbb{H} introduced by Daskalopoulos and Hamilton [3] and, independently, by Koch [22]. See [3, p. 901] for a discussion of this metric. Following [3, p. 901], we define the *cycloidal distance* between two points, $P_1 = (t_1, x^1), P_2 = (t_2, x^2) \in [0, \infty) \times \overline{\mathbb{H}}$, by

$$s(P_1, P_2) := \frac{\sum_{i=1}^d |x_i^1 - x_i^2|}{\sqrt{x_d^1} + \sqrt{x_d^2} + \sqrt{\sum_{i=1}^{d-1} |x_i^1 - x_i^2|}} + \sqrt{|t_1 - t_2|}. \quad (2.1)$$

Following [24, p. 117], we define the usual Euclidean distance between points $P_1, P_2 \in [0, \infty) \times \mathbb{R}^d$ by

$$\rho(P_1, P_2) := \sum_{i=1}^d |x_i^1 - x_i^2| + \sqrt{|t_1 - t_2|}. \quad (2.2)$$

Remark 2.1 (Equivalence of the cycloidal and Euclidean distance functions on suitable subsets of $[0, \infty) \times \mathbb{H}$). The cycloidal and Euclidean distance functions, s and ρ , are equivalent on sets of the form $[0, \infty) \times \mathbb{R}^{d-1} \times [y_0, y_1]$, for any $0 < y_0 < y_1$.

Let $\Omega \subset (0, T) \times \mathbb{H}$ be an open set and $\alpha \in (0, 1)$. We denote by $C(\bar{\Omega})$ the space of bounded, continuous functions on $\bar{\Omega}$, and by $C_0^\infty(\bar{\Omega})$ the space of smooth functions with compact support

in $\bar{\Omega}$. For a function $u : \bar{\Omega} \rightarrow \mathbb{R}$, we consider the following norms and seminorms

$$\|u\|_{C(\bar{\Omega})} = \sup_{P \in \bar{\Omega}} |u(P)|, \quad (2.3)$$

$$[u]_{C_s^\alpha(\bar{\Omega})} = \sup_{\substack{P_1, P_2 \in \bar{\Omega}, \\ P_1 \neq P_2}} \frac{|u(P_1) - u(P_2)|}{s^\alpha(P_1, P_2)}, \quad (2.4)$$

$$[u]_{C_\rho^\alpha(\bar{\Omega})} = \sup_{\substack{P_1, P_2 \in \bar{\Omega}, \\ P_1 \neq P_2}} \frac{|u(P_1) - u(P_2)|}{\rho^\alpha(P_1, P_2)}. \quad (2.5)$$

We say that $u \in C_s^\alpha(\bar{\Omega})$ if $u \in C(\bar{\Omega})$ and

$$\|u\|_{C_s^\alpha(\bar{\Omega})} = \|u\|_{C(\bar{\Omega})} + [u]_{C_s^\alpha(\bar{\Omega})} < \infty.$$

Analogously, we define the Hölder space $C_\rho^\alpha(\bar{\Omega})$ of functions u which satisfy

$$\|u\|_{C_\rho^\alpha(\bar{\Omega})} = \|u\|_{C(\bar{\Omega})} + [u]_{C_\rho^\alpha(\bar{\Omega})} < \infty.$$

We say that $u \in C_s^{2+\alpha}(\bar{\Omega})$ if

$$\|u\|_{C_s^{2+\alpha}(\bar{\Omega})} := \|u\|_{C_s^\alpha(\bar{\Omega})} + \|u_t\|_{C_s^\alpha(\bar{\Omega})} + \max_{1 \leq i \leq d} \|u_{x_i}\|_{C_s^\alpha(\bar{\Omega})} + \max_{1 \leq i, j \leq d} \|x_d u_{x_i x_j}\|_{C_s^\alpha(\bar{\Omega})} < \infty,$$

and $u \in C_\rho^{2+\alpha}(\bar{\Omega})$ if

$$\|u\|_{C_\rho^{2+\alpha}(\bar{\Omega})} = \|u\|_{C_\rho^\alpha(\bar{\Omega})} + \|u_t\|_{C_\rho^\alpha(\bar{\Omega})} + \max_{1 \leq i \leq d} \|u_{x_i}\|_{C_\rho^\alpha(\bar{\Omega})} + \max_{1 \leq i, j \leq d} \|u_{x_i x_j}\|_{C_\rho^\alpha(\bar{\Omega})} < \infty.$$

We denote by $C_{s, \text{loc}}^\alpha(\bar{\Omega})$ the space of functions u with the property that for any compact set $K \subseteq \bar{\Omega}$, we have $u \in C_s^\alpha(K)$. Analogously, we define the spaces $C_{s, \text{loc}}^{2+\alpha}(\bar{\Omega})$, $C_{\rho, \text{loc}}^\alpha(\bar{\Omega})$ and $C_{\rho, \text{loc}}^{2+\alpha}(\bar{\Omega})$.

We prove existence, uniqueness and regularity of solutions for a parabolic operator (1.2) whose second order coefficients are degenerate on $\partial\mathbb{H}$. For this purpose, we will make use of the following Hölder spaces

$$\begin{aligned} \mathcal{C}^\alpha(\bar{\mathbb{H}}_T) &:= \{u : u \in C_s^\alpha(\bar{\mathbb{H}}_{1,T}) \cap C_\rho^\alpha(\bar{\mathbb{H}}_T \setminus \mathbb{H}_{1,T})\}, \\ \mathcal{C}^{2+\alpha}(\bar{\mathbb{H}}_T) &:= \{u : u \in C_s^{2+\alpha}(\bar{\mathbb{H}}_{1,T}) \cap C_\rho^{2+\alpha}(\bar{\mathbb{H}}_T \setminus \mathbb{H}_{1,T})\}. \end{aligned}$$

We define $\mathcal{C}^\alpha(\bar{\mathbb{H}})$ and $\mathcal{C}^{2+\alpha}(\bar{\mathbb{H}})$ in the analogous manner.

The coefficient functions $x_d a_{ij}(t, x)$, $b_i(t, x)$ and $c(t, x)$ of the parabolic operator (1.2) are allowed to have linear growth in $|x|$. To account for the unboundedness of the coefficients, we augment our definition of Hölder spaces by introducing weights of the form $(1 + |x|)^q$, where $q \geq 0$ will be suitably chosen in the sequel. For $q \geq 0$, we define

$$\|u\|_{\mathcal{C}_q^0(\bar{\mathbb{H}})} := \sup_{x \in \bar{\mathbb{H}}} (1 + |x|)^q |u(x)|, \quad (2.6)$$

and, given $T > 0$, we define

$$\|u\|_{\mathcal{C}_q^0(\bar{\mathbb{H}}_T)} := \sup_{(t, x) \in \bar{\mathbb{H}}_T} (1 + |x|)^q |u(t, x)|. \quad (2.7)$$

Moreover, given $\alpha \in (0, 1)$, we define

$$\|u\|_{\mathcal{C}_q^\alpha(\bar{\mathbb{H}}_T)} := \|u\|_{\mathcal{C}_q^0(\bar{\mathbb{H}}_T)} + [(1 + |x|)^q u]_{C_s^\alpha(\bar{\mathbb{H}}_{1,T})} + [(1 + |x|)^q u]_{C_\rho^\alpha(\bar{\mathbb{H}}_T \setminus \mathbb{H}_{1,T})}, \quad (2.8)$$

$$\|u\|_{\mathcal{C}_q^{2+\alpha}(\bar{\mathbb{H}}_T)} := \|u\|_{\mathcal{C}_q^\alpha(\bar{\mathbb{H}}_T)} + \|u_t\|_{\mathcal{C}_q^\alpha(\bar{\mathbb{H}}_T)} + \|u_{x_i}\|_{\mathcal{C}_q^\alpha(\bar{\mathbb{H}}_T)} + \|x_d u_{x_i x_j}\|_{\mathcal{C}_q^\alpha(\bar{\mathbb{H}}_T)}. \quad (2.9)$$

The vector spaces

$$\begin{aligned}\mathcal{C}_q^0(\overline{\mathbb{H}}_T) &:= \left\{ u \in C(\overline{\mathbb{H}}_T) : \|u\|_{\mathcal{C}_q^0(\overline{\mathbb{H}}_T)} < \infty \right\}, \\ \mathcal{C}_q^\alpha(\overline{\mathbb{H}}_T) &:= \left\{ u \in \mathcal{C}^\alpha(\overline{\mathbb{H}}_T) : \|u\|_{\mathcal{C}_q^\alpha(\overline{\mathbb{H}}_T)} < \infty \right\}, \\ \mathcal{C}_q^{2+\alpha}(\overline{\mathbb{H}}_T) &:= \left\{ u \in \mathcal{C}^{2+\alpha}(\overline{\mathbb{H}}_T) : \|u\|_{\mathcal{C}_q^{2+\alpha}(\overline{\mathbb{H}}_T)} < \infty \right\},\end{aligned}$$

can be shown to be Banach spaces with respect to the norms (2.7), (2.8) and (2.9), respectively. We define the vector spaces $\mathcal{C}_q^0(\overline{\mathbb{H}})$, $\mathcal{C}_q^\alpha(\overline{\mathbb{H}})$, and $\mathcal{C}_q^{2+\alpha}(\overline{\mathbb{H}})$ similarly, and each can be shown to be a Banach space when equipped with the corresponding norm.

We let $\mathcal{C}_{q,\text{loc}}^{2+\alpha}(\overline{\mathbb{H}}_T)$ denote the vector space of functions u such that for any compact set $K \subset \overline{\mathbb{H}}_T$, we have $u \in \mathcal{C}_{q,\text{loc}}^{2+\alpha}(K)$, for all $q \geq 0$.

When $q = 0$, the subscript q is omitted in the preceding definitions.

2.2. Coefficients of the differential operators. Unless other conditions are explicitly substituted, we require in this article that the coefficients (a, b, c) of the parabolic differential operator L in (1.2) satisfy the conditions in the following

Assumption 2.2 (Properties of the coefficients of the parabolic differential operator). There are constants $\delta > 0$, $K > 0$, $\nu > 0$ and $\alpha \in (0, 1)$ such that the following hold.

- (1) The coefficients c and b_d obey

$$c(t, x) \leq K, \quad \forall (t, x) \in \overline{\mathbb{H}}_\infty, \quad (2.10)$$

$$b_d(t, x', 0) \geq \nu, \quad \forall (t, x') \in [0, \infty) \times \mathbb{R}^{d-1}. \quad (2.11)$$

- (2) On $\overline{\mathbb{H}}_{2,\infty}$ (that is, near $x_d = 0$), we require that

$$\sum_{i,j=1}^d a_{ij}(t, x) \eta_i \eta_j \geq \delta |\eta|^2, \quad \forall \eta \in \mathbb{R}^d, \quad \forall (t, x) \in \overline{\mathbb{H}}_{2,\infty}, \quad (2.12)$$

$$\max_{1 \leq i, j \leq d} \|a_{ij}\|_{C(\overline{\mathbb{H}}_{2,\infty})} + \max_{1 \leq i \leq d} \|b_i\|_{C(\overline{\mathbb{H}}_{2,\infty})} + \|c\|_{C(\overline{\mathbb{H}}_{2,\infty})} \leq K, \quad (2.13)$$

and, for all $P_1, P_2 \in \overline{\mathbb{H}}_{2,\infty}$ such that $P_1 \neq P_2$ and $s(P_1, P_2) \leq 1$,

$$\begin{aligned}\max_{1 \leq i, j \leq d} \frac{|a_{ij}(P_1) - a_{ij}(P_2)|}{s^\alpha(P_1, P_2)} &\leq K, \\ \max_{1 \leq i \leq d} \frac{|b_i(P_1) - b_i(P_2)|}{s^\alpha(P_1, P_2)} &\leq K, \\ \frac{|c(P_1) - c(P_2)|}{s^\alpha(P_1, P_2)} &\leq K.\end{aligned} \quad (2.14)$$

- (3) On $\overline{\mathbb{H}}_\infty \setminus \mathbb{H}_{2,\infty}$ (that is, farther away from $x_d = 0$), we require that

$$\sum_{i,j=1}^d x_d a_{ij}(t, x) \eta_i \eta_j \geq \delta |\eta|^2, \quad \forall \eta \in \mathbb{R}^d, \quad \forall (t, x) \in \overline{\mathbb{H}}_\infty \setminus \mathbb{H}_{2,\infty}, \quad (2.15)$$

and, for all $P_1, P_2 \in \overline{\mathbb{H}}_\infty \setminus \mathbb{H}_{2,\infty}$ such that $P_1 \neq P_2$ and $\rho(P_1, P_2) \leq 1$,

$$\begin{aligned} \max_{1 \leq i, j \leq d} \frac{|x_d^1 a_{ij}(P_1) - x_d^2 a_{ij}(P_2)|}{\rho^\alpha(P_1, P_2)} &\leq K, \\ \max_{1 \leq i \leq d} \frac{|b_i(P_1) - b_i(P_2)|}{\rho^\alpha(P_1, P_2)} &\leq K, \\ \frac{|c(P_1) - c(P_2)|}{\rho^\alpha(P_1, P_2)} &\leq K. \end{aligned} \quad (2.16)$$

Remark 2.3 (Local Hölder conditions on the coefficients). The local Hölder conditions (2.14) and (2.16) are similar to those in [25, Hypothesis 2.1].

Remark 2.4 (Linear growth of the coefficients of the parabolic differential operator). Conditions (2.13) and (2.16) imply that the coefficients $x_d a_{ij}(t, x)$, $b_i(t, x)$ and $c(t, x)$ can have at most linear growth in x . In particular, we may choose the constant K large enough such that

$$\sum_{i,j=1}^d |x_d a_{ij}(t, x)| + \sum_{i=1}^d |b_i(t, x)| + |c(t, x)| \leq K(1 + |x|), \quad \forall (t, x) \in \overline{\mathbb{H}}_\infty. \quad (2.17)$$

3. EXISTENCE, UNIQUENESS AND REGULARITY OF THE INHOMOGENEOUS INITIAL VALUE PROBLEM

In this section, we prove Theorem 1.1. We begin by reviewing the boundary properties and establishing the interpolation inequalities (Lemma 3.2) suitable of functions in $C_s^{2+\alpha}(\overline{\mathbb{H}}_T)$. Then, we prove two versions of the maximum principle (Proposition 3.7) which combined with the a priori local Hölder estimates at the boundary (Theorem 3.8) and in the interior (Proposition 3.14) allow us to obtain Theorem 1.1.

3.1. Boundary properties of functions in Daskalopoulos-Hamilton-Koch Hölder spaces.

The following result was proved as [3, Proposition I.12.1] when $d = 2$ and the proof when $d \geq 2$ follows by a similar argument; we include a proof for the cases omitted in [3, Proposition I.12.1].

Lemma 3.1 (Boundary properties of functions in Daskalopoulos-Hamilton-Koch Hölder spaces). [3, Proposition I.12.1] *Let $u \in C_{s,\text{loc}}^{2+\alpha}(\overline{\mathbb{H}}_T)$. Then, for all $\bar{P} \in [0, T] \times \partial\mathbb{H}$,*

$$\lim_{\overline{\mathbb{H}}_T \ni P \rightarrow \bar{P}} x_d u_{x_i x_j}(P) = 0, \quad \forall i, j = 1, \dots, d. \quad (3.1)$$

Proof. First, we consider the case $1 \leq i, j \leq d - 1$. Because the seminorm $[x_d u_{x_i x_j}]_{C_{s,\text{loc}}^\alpha(\overline{\mathbb{H}}_T)}$ is finite, the function $x_d u_{x_i x_j}$ is uniformly continuous on compact subsets of $\overline{\mathbb{H}}_T$, and so, the limit in (3.1) exists. We assume, to obtain a contradiction, that

$$\lim_{\overline{\mathbb{H}}_T \ni P \rightarrow \bar{P}} x_d u_{x_i x_j}(P) = a \neq 0, \quad (3.2)$$

and we can further assume, without loss of generality, that this limit is positive. Then, there is a constant, $\varepsilon > 0$, such that for all $P = (t, x', x_d) \in \overline{\mathbb{H}}_T$ satisfying

$$0 < x_d < \varepsilon, \quad |t - \bar{t}| < \varepsilon, \quad |x' - \bar{x}'| < \varepsilon, \quad (3.3)$$

we have

$$\frac{a}{2x_d} \leq u_{x_i x_j}(t, x', x_d). \quad (3.4)$$

Let $P_1 = (t, x^1)$ and $P_2 = (t, x^2)$ be points satisfying (3.3) and such that all except the x_i -coordinates are identical. Then, by integrating (3.4) with respect to x_i , we obtain

$$\frac{a(x_i^2 - x_i^1)}{2x_d} \leq u_{x_j}(P_2) - u_{x_j}(P_1),$$

and thus,

$$\frac{a(x_i^2 - x_i^1)}{2x_d s^\alpha(P_1, P_2)} \leq \frac{u_{x_j}(P_2) - u_{x_j}(P_1)}{s^\alpha(P_1, P_2)}. \quad (3.5)$$

We can choose P_1, P_2 such that $x_i^2 - x_i^1 = \varepsilon/2$, for all $0 < x_d < \varepsilon/2$. Then, by taking limit as x_d goes to zero, the left hand side of (3.5) diverges, while the right hand side is finite since $[u_{x_j}]_{C_s^\alpha(\mathbb{H}_T)}$ is bounded. This contradicts (3.2) and so (3.1) holds.

The case where $i = d$ or $j = d$ can be treated as in the proof of [3, Proposition I.12.1]. \square

Next, we establish the analogue of [24, Theorem 8.8.1] for the Hölder space $C_s^{2+\alpha}(\overline{\mathbb{H}}_T)$.

Lemma 3.2 (Interpolation inequalities for Daskalopoulos-Hamilton-Koch Hölder spaces). *Let $R > 0$. Then there are positive constants $m = m(d, \alpha)$ and $C = C(T, R, d, \alpha)$ such that for any $u \in C_s^{2+\alpha}(\overline{\mathbb{H}}_T)$ with compact support in $[0, \infty) \times \bar{B}_R(x^0)$, for some $x^0 \in \partial\mathbb{H}$, and any $\varepsilon \in (0, 1)$, we have*

$$\|u\|_{C_s^\alpha(\overline{\mathbb{H}}_T)} \leq \varepsilon \|u\|_{C_s^{2+\alpha}(\overline{\mathbb{H}}_T)} + C\varepsilon^{-m} \|u\|_{C(\overline{\mathbb{H}}_T)}, \quad (3.6)$$

$$\|u_{x_i}\|_{C(\overline{\mathbb{H}}_T)} \leq \varepsilon \|u\|_{C_s^{2+\alpha}(\overline{\mathbb{H}}_T)} + C\varepsilon^{-m} \|u\|_{C(\overline{\mathbb{H}}_T)}, \quad (3.7)$$

$$\|x_d u_{x_i}\|_{C_s^\alpha(\overline{\mathbb{H}}_T)} \leq \varepsilon \|u\|_{C_s^{2+\alpha}(\overline{\mathbb{H}}_T)} + C\varepsilon^{-m} \|u\|_{C(\overline{\mathbb{H}}_T)}, \quad (3.8)$$

$$\|x_d u_{x_i x_j}\|_{C(\overline{\mathbb{H}}_T)} \leq \varepsilon \|u\|_{C_s^{2+\alpha}(\overline{\mathbb{H}}_T)} + C\varepsilon^{-m} \|u\|_{C(\overline{\mathbb{H}}_T)}. \quad (3.9)$$

Remark 3.3. Notice that Lemma 3.2 does not establish the analogue of [24, Inequality (8.8.4)], that is,

$$[u_{x_i}]_{C_\rho^\alpha(\overline{\mathbb{H}}_T)} \leq \varepsilon \|u\|_{C_\rho^{2+\alpha}(\overline{\mathbb{H}}_T)} + C\varepsilon^{-m} \|u\|_{C(\overline{\mathbb{H}}_T)}.$$

This is replaced by the weighted inequality (3.8).

Proof of Lemma 3.2. We consider $\eta \in (0, 1)$, to be suitably chosen during the proofs of each of the desired inequalities.

Step 1 (Proof of inequality (3.6)). We only need to show that the first inequality (3.6) holds for the seminorm $[u]_{C_s^\alpha(\overline{\mathbb{H}}_T)}$. It is enough to consider differences, $u(P_1) - u(P_2)$, where all except one of the coordinates of the points $P_1, P_2 \in \overline{\mathbb{H}}_T$ are identical. We outline the proof when the x_i -coordinates of P_1 and P_2 differ, but the case of the t -coordinate can be treated in a similar manner. We consider two situations: $|x_i^1 - x_i^2| \leq \eta$ and $|x_i^1 - x_i^2| > \eta$.

Case 1 (Points with x_i -coordinates close together). Assume $|x_i^1 - x_i^2| \leq \eta$. We have

$$\begin{aligned} |u(P_1) - u(P_2)| &\leq |x_i^1 - x_i^2| \|u_{x_i}\|_{C(\overline{\mathbb{H}}_T)} \\ &\leq \eta \frac{|x_i^1 - x_i^2|}{\eta} \|u\|_{C_s^{2+\alpha}(\overline{\mathbb{H}}_T)} \\ &\leq \eta \left(\frac{|x_i^1 - x_i^2|}{\eta} \right)^\alpha \|u\|_{C_s^{2+\alpha}(\overline{\mathbb{H}}_T)} \\ &\leq \eta^{1-\alpha} \left(2\sqrt{x_d} + \sqrt{|x_i^1 - x_i^2|} \right)^\alpha s^\alpha(P_1, P_2) \|u\|_{C_s^{2+\alpha}(\overline{\mathbb{H}}_T)}, \end{aligned} \quad (3.10)$$

where in the last line we used the fact that, by (2.1),

$$s(P_1, P_2) = \frac{|x_i^1 - x_i^2|}{2\sqrt{x_d} + \sqrt{|x_i^1 - x_i^2|}}. \quad (3.11)$$

Because u has compact support in the spatial variable, we obtain in (3.10) that there exists a positive constant $C = C(\alpha, R)$ such that

$$\frac{|u(P_1) - u(P_2)|}{s^\alpha(P_1, P_2)} \leq C\eta^{1-\alpha}\|u\|_{C_s^{2+\alpha}(\mathbb{H}_T)}, \quad (3.12)$$

which concludes this case.

Case 2 (Points with x_i -coordinates further apart). Assume $|x_i^1 - x_i^2| > \eta$. By (3.11), we have

$$1 < \left(\frac{|x_i^1 - x_i^2|}{\eta} \right)^\alpha = \eta^{-\alpha} \left(2\sqrt{x_d} + \sqrt{|x_i^1 - x_i^2|} \right)^\alpha s^\alpha(P_1, P_2).$$

Because it suffices to consider points P_1 and P_2 in the support of u , there is a positive constant C , depending at most on α and R , such that

$$1 \leq C\eta^{-\alpha}s^\alpha(P_1, P_2).$$

Therefore,

$$|u(P_1) - u(P_2)| \leq 2\|u\|_{C(\mathbb{H}_T)} \leq C\eta^{-\alpha}s^\alpha(P_1, P_2)\|u\|_{C(\mathbb{H}_T)},$$

which is equivalent to

$$\frac{|u(P_1) - u(P_2)|}{s^\alpha(P_1, P_2)} \leq C\eta^{-\alpha}\|u\|_{C^0(\mathbb{H}_T)}, \quad (3.13)$$

which concludes this case.

By combining (3.12) and (3.13), we obtain

$$[u]_{C_s^\alpha(\mathbb{H}_T)} \leq C\eta^{1-\alpha}\|u\|_{C_s^{2+\alpha}(\mathbb{H}_T)} + C\eta^{-\alpha}\|u\|_{C^0(\mathbb{H}_T)}.$$

Since $\varepsilon \in (0, 1)$, we may choose $\eta \in (0, 1)$ such that $\varepsilon = C\eta^{1-\alpha}$. The preceding inequality then gives (3.6).

Step 2 (Proof of inequality (3.7)). Let $P \in \overline{\mathbb{H}}_T$. Then, for any $\eta > 0$, we have

$$\begin{aligned} |u_{x_i}(P)| &\leq |u_{x_i}(P) - \eta^{-1}(u(P + \eta e_i) - u(P))| + 2\eta^{-1}\|u\|_{C(\mathbb{H}_T)} \\ &= |u_{x_i}(P) - u_{x_i}(P + \eta\theta e_i)| + 2\eta^{-1}\|u\|_{C(\mathbb{H}_T)} \\ &= \frac{|u_{x_i}(P) - u_{x_i}(P + \eta\theta e_i)|}{s(P, P + \eta\theta e_i)^\alpha} s^\alpha(P, P + \eta\theta e_i) + 2\eta^{-1}\|u\|_{C(\mathbb{H}_T)}, \end{aligned}$$

for some constant $\theta \in [0, 1]$. Using

$$s(P, P + \eta\theta e_i) \leq \eta^{1/2}, \quad \forall P \in \overline{\mathbb{H}}_T, \quad (3.14)$$

we have

$$|u_{x_i}(P)| \leq \eta^{\alpha/2}[u_{x_i}]_{C_s^\alpha(\mathbb{H}_T)} + 2\eta^{-1}\|u\|_{C(\mathbb{H}_T)}, \quad \forall P \in \overline{\mathbb{H}}_T. \quad (3.15)$$

Since $\varepsilon \in (0, 1)$, we may choose $\eta \in (0, 1)$ such that $\varepsilon = \eta^{\alpha/2}$. Then (3.7) follows from (3.15).

Step 3 (Proof of inequality (3.8)). Because u has compact support in the spatial variable, then (3.7) gives, for some positive constant $C = C(\alpha, R)$,

$$\|x_d u_{x_i}\|_{C(\overline{\mathbb{H}}_T)} \leq C\varepsilon \|u\|_{C_s^{2+\alpha}(\overline{\mathbb{H}}_T)} + C\varepsilon^{-m} \|u\|_{C(\overline{\mathbb{H}}_T)}. \quad (3.16)$$

This gives the desired bound in (3.8) for the term $\|x_d u_{x_i}\|_{C(\overline{\mathbb{H}}_T)}$. It remains to prove the estimate (3.8) for the Hölder seminorm $[x_d u_{x_i}]_{C_s^\alpha(\overline{\mathbb{H}}_T)}$. As in the proof of (3.6), it suffices to consider the differences $x_d^1 u_{x_i}(P_1) - x_d^2 u_{x_i}(P_2)$, where all except one of the coordinates of the points $P_1, P_2 \in \overline{\mathbb{H}}_T$ are identical.

First, we consider the case when only the x_d -coordinates of the points P_1 and P_2 differ. We denote $P_k = (t, x', x_d^k)$, $k = 1, 2$.

Case 1 (Points with x_d -coordinates close together). Assume $|x_d^1 - x_d^2| \leq \eta$. Using

$$(x_d u_{x_i})_{x_d} = x_d u_{x_i x_d} + u_{x_i}$$

and the mean value theorem, there is a point P^* on the line segment connecting P_1 and P_2 such that,

$$x_d^1 u_{x_i}(P_1) - x_d^2 u_{x_i}(P_2) = (x_d^* u_{x_i x_d}(P^*) + u_{x_i}(P^*)) (x_d^1 - x_d^2),$$

and so,

$$\begin{aligned} |x_d^1 u_{x_i}(P_1) - x_d^2 u_{x_i}(P_2)| &\leq \eta \left(\frac{|x_d^1 - x_d^2|}{\eta} \right)^\alpha \|u\|_{C_s^{2+\alpha}(\overline{\mathbb{H}}_T)} \\ &\leq \eta^{1-\alpha} \left(\sqrt{x_d^1} + \sqrt{x_d^2} + \sqrt{|x_d^1 - x_d^2|} \right)^\alpha s^\alpha(P_1, P_2) \|u\|_{C_s^{2+\alpha}(\overline{\mathbb{H}}_T)}. \end{aligned}$$

Because u has compact support in the spatial variable, there is a positive constant $C = C(\alpha, R)$ such that

$$\frac{|x_d^1 u_{x_i}(P_1) - x_d^2 u_{x_i}(P_2)|}{s^\alpha(P_1, P_2)} \leq C \eta^{1-\alpha} \|u\|_{C_s^{2+\alpha}(\overline{\mathbb{H}}_T)}, \quad (3.17)$$

which concludes this case.

Case 2 (Points with x_d -coordinates further apart). Assume $|x_d^1 - x_d^2| > \eta$. We have

$$\begin{aligned} \frac{|x_d^1 u_{x_i}(P_1) - x_d^2 u_{x_i}(P_2)|}{s^\alpha(P_1, P_2)} &\leq 2 \frac{\|x_d u_{x_i}\|_{C(\overline{\mathbb{H}}_T)}}{|x_d^1 - x_d^2|^\alpha} \left(\sqrt{x_d^1} + \sqrt{x_d^2} + \sqrt{|x_d^1 - x_d^2|} \right)^\alpha \\ &\leq C \eta^{-\alpha} \|x_d u_{x_i}\|_{C(\overline{\mathbb{H}}_T)}. \end{aligned}$$

Since $\varepsilon \in (0, 1)$, we may choose η such that $\varepsilon = \eta^{\alpha+1}$ in (3.16). We obtain

$$\frac{|x_d^1 u_{x_i}(P_1) - x_d^2 u_{x_i}(P_2)|}{s^\alpha(P_1, P_2)} \leq C \eta \|u\|_{C_s^{2+\alpha}(\overline{\mathbb{H}}_T)} + C \eta^{-m(1+\alpha)-\alpha} \|u\|_{C(\overline{\mathbb{H}}_T)}, \quad (3.18)$$

which concludes this case.

Combining (3.17) and (3.18) gives

$$\frac{|x_d^1 u_{x_i}(P_1) - x_d^2 u_{x_i}(P_2)|}{s^\alpha(P_1, P_2)} \leq C \eta^{1-\alpha} \|u\|_{C_s^{2+\alpha}(\overline{\mathbb{H}}_T)} + C \eta^{-m(1+\alpha)-\alpha} \|u\|_{C(\overline{\mathbb{H}}_T)}. \quad (3.19)$$

A similar argument, when only the x_i -coordinates of the points P_1 and P_2 differ, $1 \leq i \leq d-1$, also yields (3.19).

Next, we consider the case when only the t -coordinates of the points P_1 and P_2 differ. We denote $P_k = (x, t_k)$, $k = 1, 2$. We shall only describe the proof of the interpolation inequality for u_{x_i} when $i \neq d$, as the case $i = d$ follows by a similar argument. We denote $\delta = \sqrt{|t_1 - t_2|}$.

Case 1 (Points with t -coordinates close together). Assume $|t_1 - t_2| < \eta$. We have

$$\begin{aligned} |u_{x_i}(P_1) - u_{x_i}(P_2)| &\leq \left| u_{x_i}(x, t_1) - \frac{1}{\delta} (u(x + \delta e_i, t_1) - u(x, t_1)) \right| \\ &\quad + \left| u_{x_i}(x, t_2) - \frac{1}{\delta} (u(x + \delta e_i, t_2) - u(x, t_2)) \right| \\ &\quad + \frac{1}{\delta} |u(x + \delta e_i, t_1) - u(x + \delta e_i, t_2)| + \frac{1}{\delta} |u(x, t_1) - u(x, t_2)|. \end{aligned}$$

By the mean value theorem, there are points $P_k^* \in \overline{\mathbb{H}}_T$, $k = 1, 2$, such that

$$\begin{aligned} |u_{x_i}(P_1) - u_{x_i}(P_2)| &= |u_{x_i}(x, t_1) - u_{x_i}(x + \theta_1 \delta e_i, t_1)| + |u_{x_i}(x, t_2) - u_{x_i}(x + \theta_2 \delta e_i, t_2)| \\ &\quad + \frac{|t_1 - t_2|}{\delta} |u_t(x + \delta e_i, t_1^*)| + \frac{|t_1 - t_2|}{\delta} |u_t(x, t_2^*)| \\ &\leq |u_{x_i x_i}(P_1^*, t_1)| \delta + |u_{x_i x_i}(P_2^*, t_2)| \delta \\ &\quad + \frac{|t_1 - t_2|}{\delta} |u_t(x + \delta e_i, t_1^*)| + \frac{|t_1 - t_2|}{\delta} |u_t(x, t_2^*)|. \end{aligned}$$

Notice that $s(P_1, P_2) = \sqrt{|t_1 - t_2|} = \delta$ and so, by multiplying the preceding inequality by x_d and using the fact that u has compact support, we obtain

$$\frac{|x_d u_{x_i}(P_1) - x_d u_{x_i}(P_2)|}{s^\alpha(P_1, P_2)} \leq 2 \|x_d u_{x_i x_i}\|_{C^0(\overline{\mathbb{H}}_T)} |t_1 - t_2|^{\frac{1-\alpha}{2}} + 2 |t_1 - t_2|^{1-\frac{1+\alpha}{2}} \|x_d u_t\|_{C^0(\overline{\mathbb{H}}_T)},$$

and thus

$$\frac{|x_d u_{x_i}(P_1) - x_d u_{x_i}(P_2)|}{s^\alpha(P_1, P_2)} \leq C \eta^{\frac{1-\alpha}{2}} \|u\|_{C_s^{2+\alpha}(\overline{\mathbb{H}}_T)}, \quad (3.20)$$

where C is a positive constant depending only on R .

Case 2 (Points with t -coordinates further apart). Assume $|t_1 - t_2| \geq \eta$. This case is easier, as usual, because

$$\frac{|x_d u_{x_i}(P_1) - x_d u_{x_i}(P_2)|}{s^\alpha(P_1, P_2)} \leq 2 \eta^{-\frac{\alpha}{2}} \|x_d u_{x_i}\|_{C(\overline{\mathbb{H}}_T)}, \quad (3.21)$$

which concludes this case.

By combining inequalities (3.20) and (3.21), we obtain

$$\frac{|x_d u_{x_i}(P_1) - x_d u_{x_i}(P_2)|}{s^\alpha(P_1, P_2)} \leq C \eta^{\frac{1-\alpha}{2}} \|u\|_{C_s^{2+\alpha}(\overline{\mathbb{H}}_T)} + 2 \eta^{-\frac{\alpha}{2}} \|x_d u_{x_i}\|_{C(\overline{\mathbb{H}}_T)}. \quad (3.22)$$

By (3.19) and (3.22), we have

$$[x_d u_{x_i}]_{C_s^{\alpha_0}(\overline{\mathbb{H}}_T)} \leq C \eta^{\alpha_0} \|u\|_{C_s^{2+\alpha}(\overline{\mathbb{H}}_T)} + 2 \eta^{-m_0} \|x_d u_{x_i}\|_{C(\overline{\mathbb{H}}_T)},$$

where $\alpha_0 := \min\{\alpha, 1 - \alpha, (1 - \alpha)/2\}$ and $m_0 := 4 + \alpha$. Without loss of generality, we may assume $C \geq 1$. Since $\varepsilon \in (0, 1)$, we may choose $\eta \in (0, 1)$ such that $\varepsilon = C \eta^{\alpha_0}$ in the preceding inequality, and so we obtain the estimate (3.8) for $[x_d u_{x_i}]_{C_s^{\alpha_0}(\overline{\mathbb{H}}_T)}$. This concludes the proof of (3.8).

Step 4 (Proof of inequality (3.9)). For any $P = (t, x) \in \overline{\mathbb{H}}_T$, we can find $\theta \in [0, 1]$ such that

$$|x_d u_{x_i x_j}(P)| \leq |x_d u_{x_i x_j}(P) - (x_d u_{x_i}(P + \eta e_j) - x_d u_{x_i}(P))| + 2 \|x_d u_{x_i}\|_{C(\overline{\mathbb{H}}_T)},$$

and thus

$$|x_d u_{x_i x_j}(P)| \leq |x_d u_{x_i x_j}(P) - x_d u_{x_i x_j}(P + \theta \eta e_j)| + 2 \|x_d u_{x_i}\|_{C(\overline{\mathbb{H}}_T)}, \quad (3.23)$$

where $1 \leq i, j \leq d$. If $j \neq d$, we have

$$\begin{aligned} |x_d u_{x_i x_j}(P)| &\leq \frac{|x_d u_{x_i x_j}(P) - x_d u_{x_i x_j}(P + \theta \eta e_j)|}{s^\alpha(P, P + \theta \varepsilon e_j)} s^\alpha(P, P + \theta \eta e_j) + 2 \|x_d u_{x_i}\|_{C(\overline{\mathbb{H}}_T)} \\ &\leq C \eta^{\alpha/2} [x_d u_{x_i x_j}]_{C_s^\alpha(\overline{\mathbb{H}}_T)} + 2 \|x_d u_{x_i}\|_{C(\overline{\mathbb{H}}_T)}, \quad (\text{by (3.14)}). \end{aligned}$$

Because $\varepsilon \in (0, 1)$, we may choose $\eta \in (0, 1)$ such that $\varepsilon = C \eta^{\alpha/2}$ in the preceding inequality and combining the resulting inequality with (3.8), we see that the estimate (3.9) for $\|x_d u_{x_i x_j}\|_{C(\overline{\mathbb{H}}_T)}$ holds for all $j \neq d$.

Next, we consider the case $j = d$. For brevity, we denote $P' = P + \theta \eta e_d = (t, x', x'_d)$ and $P'' = (t, x', 0)$. We consider two distinct cases depending on whether $\eta < x'_d/2$ or $\eta \geq x'_d/2$.

Case 1 (Points with x_d -coordinates further apart). Assume $\eta < x'_d/2$. By (3.23), we obtain

$$\begin{aligned} |x_d u_{x_i x_d}(P)| &\leq \frac{|x_d u_{x_i x_d}(P) - x'_d u_{x_i x_d}(P')|}{s^\alpha(P, P')} s^\alpha(P, P') \\ &\quad + |(x'_d - x_d) u_{x_i x_d}(P')| + 2 \|x_d u_{x_i}\|_{C(\overline{\mathbb{H}}_T)}, \end{aligned} \quad (3.24)$$

and so, using (3.14) and the fact that $|x'_d - x_d| \leq \eta$, by definitions of points P and P' ,

$$|x_d u_{x_i x_d}(P)| \leq \eta^{\alpha/2} [x_d u_{x_i x_d}]_{C_s^\alpha(\overline{\mathbb{H}}_T)} + \frac{\eta}{x'_d} |x'_d u_{x_i x_d}(P')| + 2 \|x_d u_{x_i}\|_{C(\overline{\mathbb{H}}_T)},$$

which gives, by our assumption that $\eta < x'_d/2$,

$$|x_d u_{x_i x_d}(P)| \leq \eta^{\alpha/2} [x_d u_{x_i x_d}]_{C_s^\alpha(\overline{\mathbb{H}}_T)} + \frac{1}{2} \|x_d u_{x_i x_d}\|_{C(\overline{\mathbb{H}}_T)} + 2 \|x_d u_{x_i}\|_{C(\overline{\mathbb{H}}_T)}. \quad (3.25)$$

As (3.25) holds for all $P \in \overline{\mathbb{H}}_T$, we have

$$\|x_d u_{x_i x_d}\|_{C(\overline{\mathbb{H}}_T)} \leq \frac{1}{2} \|x_d u_{x_i x_d}\|_{C(\overline{\mathbb{H}}_T)} + \eta^{\alpha/2} [x_d u_{x_i x_d}]_{C_s^\alpha(\overline{\mathbb{H}}_T)} + 2 \|x_d u_{x_i}\|_{C(\overline{\mathbb{H}}_T)},$$

or

$$\|x_d u_{x_i x_d}\|_{C(\overline{\mathbb{H}}_T)} \leq 2 \eta^{\alpha/2} [x_d u_{x_i x_d}]_{C_s^\alpha(\overline{\mathbb{H}}_T)} + 4 \|x_d u_{x_i}\|_{C(\overline{\mathbb{H}}_T)}, \quad (3.26)$$

which concludes this case.

Case 2 (Points with x_d -coordinates close together). Assume $\eta \geq x'_d/2$. Recall that $x'_d = x_d + \theta \eta$, for some $\theta \in [0, 1]$, so that $|x'_d - x_d| \leq x'_d$. From Lemma 3.1, we have

$$x_d u_{x_i x_d} \rightarrow 0, \text{ as } x_d \rightarrow 0.$$

Therefore, we obtain

$$\begin{aligned} |(x'_d - x_d) u_{x_i x_d}(P')| &\leq |x'_d u_{x_i x_d}(P')| = \frac{|x'_d u_{x_i x_d}(P') - 0|}{s^\alpha(P', P'')} s^\alpha(P', P'') \\ &\leq [x_d u_{x_i x_d}]_{C_s^\alpha(\overline{\mathbb{H}}_T)} (2\eta)^{\alpha/2}, \end{aligned}$$

where the second inequality follows from the fact that

$$s(P', P'') \leq \sqrt{x'_d} \leq \sqrt{2\eta}.$$

By a calculation similar to that which led to (3.24), we obtain

$$\begin{aligned} |x_d u_{x_i x_d}(P)| &\leq \frac{|x_d u_{x_i x_d}(P) - x'_d u_{x_i x_d}(P')|}{s^\alpha(P, P')} s^\alpha(P, P') \\ &\quad + |(x'_d - x_d) u_{x_i x_d}(P')| + 2 \|x_d u_{x_i}\|_{C(\overline{\mathbb{H}}_T)}, \end{aligned}$$

and hence

$$|x_d u_{x_i x_d}(P)| \leq C\eta^{\alpha/2} [x_d u_{x_i x_d}]_{C_s^\alpha(\overline{\mathbb{H}}_T)} + (2\eta)^{\alpha/2} [x_d u_{x_i x_d}]_{C_s^\alpha(\overline{\mathbb{H}}_T)} + 2\|x_d u_{x_i}\|_{C(\overline{\mathbb{H}}_T)}, \quad (3.27)$$

which concludes this case.

By combining inequalities (3.25) and (3.27), we obtain, for all $P \in \overline{\mathbb{H}}_T$,

$$|x_d u_{x_i x_d}(P)| \leq \frac{1}{2}\|x_d u_{x_i x_d}\|_{C(\overline{\mathbb{H}}_T)} + C\eta^{\alpha/2} [x_d u_{x_i x_d}]_{C_s^\alpha(\overline{\mathbb{H}}_T)} + 2\|x_d u_{x_i}\|_{C(\overline{\mathbb{H}}_T)},$$

which is equivalent to

$$\|x_d u_{x_i x_d}\|_{C(\overline{\mathbb{H}}_T)} \leq \frac{1}{2}\|x_d u_{x_i x_d}\|_{C(\overline{\mathbb{H}}_T)} + C\eta^{\alpha/2} [x_d u_{x_i x_d}]_{C_s^\alpha(\overline{\mathbb{H}}_T)} + 2\|x_d u_{x_i}\|_{C(\overline{\mathbb{H}}_T)}.$$

Rearranging terms yields

$$\|x_d u_{x_i x_d}\|_{C(\overline{\mathbb{H}}_T)} \leq 2C\eta^{\alpha/2} [x_d u_{x_i x_d}]_{C_s^\alpha(\overline{\mathbb{H}}_T)} + 4\|x_d u_{x_i}\|_{C(\overline{\mathbb{H}}_T)}. \quad (3.28)$$

Since $\varepsilon \in (0, 1)$, we may choose $\eta \in (0, 1)$ in (3.26) and (3.28) such that $\varepsilon = 4(C + 1)\eta^{\alpha/2}$ and so we obtain

$$\|x_d u_{x_i x_d}\|_{C(\overline{\mathbb{H}}_T)} \leq \varepsilon/2 [x_d u_{x_i x_d}]_{C_s^\alpha(\overline{\mathbb{H}}_T)} + 4\|x_d u_{x_i}\|_{C(\overline{\mathbb{H}}_T)}.$$

Combining the preceding inequality with (3.8) applied with ε replaced by $\varepsilon/8$, we conclude that (3.9) holds.

This completes the proof of Lemma 3.2. \square

3.2. Maximum principle and its applications. In this subsection, we prove a variant of the classical maximum principle (see [24, Section 8.1] and [3, Theorem I.3.1]) for parabolic operators, L , of the form (1.2).

Lemma 3.4 (Maximum principle). *We relax the requirements stated in Assumption 2.2 on the coefficients $a = (a_{ij}), b = (b_i), c$ of the operator L in (1.2) to those stated here. Require that the coefficients $x_d a_{ij}, b_i, c$ be in $C_{\text{loc}}((0, T] \times \overline{\mathbb{H}})$, that $b_d \geq 0$ when $x_d = 0$, that c obeys (2.10), and*

$$\text{tr}(x_d a(t, x)) + x \cdot b(t, x) \leq K(1 + |x|^2), \quad \forall (t, x) \in \overline{\mathbb{H}}_T, \quad (3.29)$$

where $K > 0$. Suppose $u \in C^{1,2}(\mathbb{H}_T) \cap C(\overline{\mathbb{H}}_T)$ obeys

$$u_t, u_{x_i}, x_d u_{x_i x_j} \in C_{\text{loc}}((0, T] \times \overline{\mathbb{H}}), \quad 1 \leq i, j \leq d, \quad (3.30)$$

and

$$x_d u_{x_i x_j} = 0 \text{ on } (0, T] \times \partial\mathbb{H}, \quad 1 \leq i, j \leq d. \quad (3.31)$$

If

$$Lu \leq 0 \quad \text{on } (0, T) \times \mathbb{H}, \quad (3.32)$$

$$u(0, \cdot) \leq 0 \quad \text{on } \overline{\mathbb{H}}, \quad (3.33)$$

then

$$u \leq 0 \quad \text{on } [0, T] \times \overline{\mathbb{H}}. \quad (3.34)$$

Proof. We apply an argument similar to that used in the proofs of [24, Theorem 2.9.2, Exercises 2.9.4 & 2.9.5] (maximum principle for elliptic equations on unbounded domains) and [24, Theorems 8.1.2 & 8.1.4] (maximum principle for parabolic equations on unbounded domains); see also [3, Theorem I.3.1].

We consider the transformation

$$u(t, x) = e^{\lambda t} \tilde{u}(t, x) \quad \text{on } [0, T] \times \overline{\mathbb{H}}, \quad (3.35)$$

where the constant $\lambda > 0$ will be suitably chosen below. The conclusion of the lemma follows if and only if (3.34) holds for \tilde{u} . By (3.32) and definition (3.35) we have

$$e^{\lambda t} (L + \lambda) \tilde{u} = Lu \leq 0 \quad \text{on } (0, T) \times \mathbb{H}.$$

Therefore, by (3.32) and (3.33), the function \tilde{u} satisfies

$$(L + \lambda) \tilde{u} \leq 0 \quad \text{on } (0, T) \times \mathbb{H}, \quad (3.36)$$

$$\tilde{u}(0, \cdot) \leq 0 \quad \text{on } \overline{\mathbb{H}}. \quad (3.37)$$

We may suppose without loss of generality that

$$m := \sup_{\mathbb{H}_T} \tilde{u} \geq 0, \quad (3.38)$$

as if $m < 0$ we are done; we shall show that $m = 0$. Define an auxiliary function,

$$h(t, x) := 1 + |x|^2, \quad \forall (t, x) \in \overline{\mathbb{H}}_T. \quad (3.39)$$

By direct calculation,

$$\begin{aligned} -(L + \lambda) h &= \sum_{i,j=1}^d x_d a_{ij} h_{x_i x_j} + \sum_{i=1}^d b_i h_{x_i} + (c - \lambda) h - h_t \\ &= 2x_d \sum_{i=1}^d a_{ii} + 2 \sum_{i=1}^d b_i x_i + (c - \lambda)(1 + |x|^2) \\ &\leq (2K + c - \lambda)(1 + |x|^2) \quad \text{on } (0, T) \times \mathbb{H}, \quad (\text{by (3.29)}) \end{aligned}$$

By choosing

$$\lambda \geq 3K, \quad (3.40)$$

we notice that condition (2.10), gives

$$2K + c(t, x) - \lambda \leq 0 \quad \forall (t, x) \in \overline{\mathbb{H}}_T, \quad (3.41)$$

and so, we have

$$(L + \lambda) h \geq 0 \quad \text{on } (0, T) \times \mathbb{H}. \quad (3.42)$$

Fix $\delta \in (0, 1)$ and define another auxiliary function

$$w := \tilde{u} - \delta m h. \quad (3.43)$$

From (3.36) and (3.42), we have $(L + \lambda)w \leq 0$ on $(0, T) \times \mathbb{H}$ and thus

$$(L + \lambda) w \leq 0 \quad \text{on } (0, T] \times \overline{\mathbb{H}}, \quad (3.44)$$

since $w_t, w_{x_i}, x_d w_{x_i x_j}$ extend continuously from $(0, T) \times \mathbb{H}$ to $(0, T] \times \overline{\mathbb{H}}$ because these continuity properties are true of u by hypothesis (3.30) (and trivially true for h) and thus also true for w .

Claim 3.5. *There is a constant, $R_0 = R_0(\delta) > 0$, such that*

$$w \leq 0 \quad \text{on } [0, T] \times \bar{B}_R, \quad \forall R \geq R_0(\delta). \quad (3.45)$$

Proof. Since $w \in C([0, T] \times \bar{B}_R)$, the function w attains its maximum at some point $P \in [0, T] \times \bar{B}_R$. If $P \in (0, T] \times B_R$, then

$$w_t(P) \geq 0, \quad w_{x_i}(P) = 0, \quad (w_{x_i x_j}(P)) \leq 0.$$

Therefore,

$$\begin{aligned} -(L + \lambda) w(P) &= \sum_{i,j=1}^d x_d a_{ij}(P) w_{x_i x_j}(P) + \sum_{i=1}^d b_i(P) w_{x_i}(P) + (c(P) - \lambda) w(P) - w_t(P) \\ &\leq (c(P) - \lambda) w(P). \end{aligned}$$

If $P \in (0, T] \times (\bar{B}_R \cap \{x_d = 0\})$, then

$$w_t(P) \geq 0, \quad w_{x_d}(P) \leq 0, \quad w_{x_i}(P) = 0 \quad (i \neq d), \quad x_d w_{x_i x_j}(P) = 0,$$

where we use the fact that u , and thus w , obey (3.30) and (3.31). Therefore,

$$\begin{aligned} -(L + \lambda) w(P) &= \sum_{i,j=1}^d x_d a_{ij}(P) w_{x_i x_j}(P) + \sum_{i=1}^d b_i(P) w_{x_i}(P) + c(P) w(P) - w_t(P) \\ &\leq b_d(P) w_{x_d}(P) + (c(P) - \lambda) w(P) \\ &\leq (c(P) - \lambda) w(P) \quad (\text{by hypothesis that } b_d \geq 0 \text{ on } \{x_d = 0\}). \end{aligned}$$

Hence, for $P \in (0, T] \times B_R$ or $(0, T] \times (\bar{B}_R \cap \{x_d = 0\})$, we obtain

$$-(c(P) - \lambda) w(P) \leq L w(P).$$

But $L w(P) \leq 0$ by (3.44) and therefore, $w(P) \leq 0$ since $c \leq K$ by (2.10) and $\lambda \geq 3K$ by (3.40).

Now suppose P lies in one of the remaining two components of the boundary of $(0, T) \times B_R$,

$$\mathcal{B}_R^0 := \{0\} \times \bar{B}_R \quad \text{or} \quad \mathcal{B}_R^1 := (0, T] \times (\{x_d > 0\} \cap \partial B_R).$$

The definition (3.39) of h , definition (3.43) of w , and (3.37) yield

$$w(0, \cdot) \leq 0 \text{ on } \bar{B}_R, \quad \forall R > 0, \tag{3.46}$$

and thus, $w(P) \leq 0$ if $P \in \mathcal{B}_R^0$, for $R > 0$. If $P \in \mathcal{B}_R^1$, then $|x| = R$ and we see that (3.38), (3.39), and (3.43) give

$$\begin{aligned} w(P) &= \tilde{u}(P) - \delta m h(P) \\ &\leq m - \delta m(1 + R^2) \\ &= m(1 - \delta(1 + R^2)). \end{aligned}$$

But $1 - \delta(1 + R^2) \leq 0$ provided $R \geq R_0(\delta) := (\delta^{-1} - 1)^{1/2} > 0$ and so $w(P) \leq 0$ for all $R \geq R_0(\delta)$. This completes the proof of Claim 3.5. \square

By (3.45), we see that

$$w = \tilde{u} - \delta m h \leq 0 \quad \text{on } \bar{\mathbb{H}}_T,$$

for all $\delta \in (0, 1)$ and thus, letting $\delta \downarrow 0$, we obtain (3.34). \square

Lemma 3.4 immediately leads to the following comparison principle.

Corollary 3.6 (Comparison principle). *Assume that the coefficients of L in (1.2) obey the hypotheses of Lemma 3.4. If $u, v \in C^{1,2}(\mathbb{H}_T) \cap C(\overline{\mathbb{H}}_T)$ obey (3.30), (3.31), and*

$$Lu \leq Lv \quad \text{on } (0, T) \times \mathbb{H}, \quad (3.47)$$

$$u(0, \cdot) \leq v(0, \cdot) \quad \text{on } \overline{\mathbb{H}}, \quad (3.48)$$

then

$$u \leq v \quad \text{on } [0, T] \times \overline{\mathbb{H}}. \quad (3.49)$$

Note that if (3.47) and (3.48) are strengthened to

$$|Lu| \leq Lv \quad \text{on } (0, T) \times \mathbb{H} \quad \text{and} \quad |u(0, \cdot)| \leq v(0, \cdot) \quad \text{on } \overline{\mathbb{H}}, \quad (3.50)$$

then Corollary 3.6 yields

$$|u| \leq v \quad \text{on } [0, T] \times \overline{\mathbb{H}}. \quad (3.51)$$

We can now turn our attention to the

Proposition 3.7 (Application of the maximum principle). *Assume that the coefficients of L in (1.2) obey the hypotheses of Lemma 3.4, except that (3.29) is replaced by the stronger condition*

$$\sum_{i,j=1}^d x_d |a_{ij}(t, x)| + |x \cdot b(t, x)| \leq K(1 + |x|^2), \quad \forall (t, x) \in \overline{\mathbb{H}}_T. \quad (3.52)$$

Suppose that $u \in C^{1,2}(\mathbb{H}_T) \cap C(\overline{\mathbb{H}}_T)$ solves (1.1) and obeys (3.30) and (3.31).

(a) If $f \in C(\overline{\mathbb{H}}_T)$ and $g \in C(\overline{\mathbb{H}})$, then

$$\|u\|_{C(\overline{\mathbb{H}}_T)} \leq e^{KT} \left(T\|f\|_{C(\overline{\mathbb{H}}_T)} + \|g\|_{C(\overline{\mathbb{H}})} \right). \quad (3.53)$$

(b) If $q > 0$, $f \in \mathcal{C}_q^0(\overline{\mathbb{H}}_T)$, and $g \in \mathcal{C}_q^0(\overline{\mathbb{H}})$, then

$$\|u\|_{\mathcal{C}_q^0(\overline{\mathbb{H}}_T)} \leq e^{(1+q(q+4)K)T} \left(\|f\|_{\mathcal{C}_q^0(\overline{\mathbb{H}}_T)} + \|g\|_{\mathcal{C}_q^0(\overline{\mathbb{H}})} \right). \quad (3.54)$$

Proof. To obtain (3.53) and (3.54), we make specific choices of the function v in Corollary 3.6. To establish (3.53), we choose

$$v_1(t, x) := e^{Kt} \left(t\|f\|_{C(\overline{\mathbb{H}}_T)} + \|g\|_{C(\overline{\mathbb{H}})} \right), \quad \forall (t, x) \in \overline{\mathbb{H}}_T,$$

Direct calculation gives

$$\begin{aligned} Lv_1 &= (-c + K)v_1 + e^{Kt}\|f\|_{C(\overline{\mathbb{H}}_T)} \\ &\geq \|f\|_{C(\overline{\mathbb{H}}_T)} \quad \text{on } (0, T) \times \mathbb{H} \quad (\text{by (2.10)}). \end{aligned}$$

Therefore, since $Lu = f$ on $(0, T) \times \mathbb{H}$ by (1.1),

$$|Lu| \leq Lv_1 \quad \text{on } (0, T) \times \mathbb{H},$$

and so v_1 satisfies conditions (3.50). Thus, by (3.51), we obtain (3.53).

Next, we prove (3.54). For this purpose, we choose

$$v_2(t, x) := e^{\lambda t} \frac{\left(\|f\|_{\mathcal{C}_q^0(\overline{\mathbb{H}}_T)} + \|g\|_{\mathcal{C}_q^0(\overline{\mathbb{H}})} \right)}{(1 + |x|^2)^{q/2}}, \quad \forall (t, x) \in \overline{\mathbb{H}}_T, \quad (3.55)$$

where $\lambda > 0$ will be suitably chosen below. First, we verify that v_2 satisfies the first inequality in (3.50). Direct calculation gives

$$Lv_2 = v_2 \left[-c(t, x) + \lambda + q \sum_{i=1}^d \frac{b_i(t, x)x_i}{1 + |x|^2} - q(q+2) \sum_{i,j=1}^d \frac{a_{ij}(t, x)x_i x_j x_d}{(1 + |x|^2)^2} + q \sum_{i=1}^d \frac{a_{ii}(t, x)x_d}{1 + |x|^2} \right].$$

Conditions (3.52) and (2.10), imply that

$$Lv_2 \geq v_2 (K + \lambda - qK - q(q+2)K - qK).$$

By choosing

$$\lambda = 1 + q(q+4)K > 0,$$

we obtain

$$Lv_2 \geq v_2 \geq \frac{\|f\|_{\mathcal{C}_q^0(\overline{\mathbb{H}}_T)}}{(1 + |x|^2)^{q/2}} \quad \text{on } (0, T) \times \mathbb{H}.$$

By the definition (2.7) of the norm $\|\cdot\|_{\mathcal{C}_q^0(\overline{\mathbb{H}}_T)}$, we have

$$(1 + |x|^2)^{q/2} |f(t, x)| \leq \|f\|_{\mathcal{C}_q^0(\overline{\mathbb{H}}_T)}, \quad \forall (t, x) \in [0, T] \times \overline{\mathbb{H}},$$

and so, using $Lu = f$ on \mathbb{H}_T by (1.1), we obtain the first inequality in (3.50), that is,

$$|Lu| \leq Lv_2 \quad \text{on } (0, T) \times \mathbb{H}. \quad (3.56)$$

Similarly, by the definition (2.6) of the norm $\|\cdot\|_{\mathcal{C}_q^0(\overline{\mathbb{H}})}$, we have

$$(1 + |x|^2)^{q/2} |g(x)| \leq \|g\|_{\mathcal{C}_q^0(\overline{\mathbb{H}})}, \quad \forall x \in \overline{\mathbb{H}}.$$

Since $u(0, \cdot) = g$ on $\overline{\mathbb{H}}$, it is immediate that

$$|u(0, \cdot)| \leq v_2(0, \cdot) \quad \text{on } \overline{\mathbb{H}}. \quad (3.57)$$

Therefore, by (3.56) and (3.57), v_2 obeys conditions (3.50), and so we obtain (3.54) from the definition (3.55) of v_2 . \square

3.3. Local a priori boundary estimates. We have the following analogue of [24, Theorem 8.11.1].

Theorem 3.8 (A priori boundary estimates). *There is constant $R^* = R^*(d, \alpha, K, \delta, \nu)$, such that for any $0 < R \leq R^*$, we can find a positive constant $C = C(d, \alpha, K, \delta, \nu, R)$, such that for any $x^0 \in \partial\mathbb{H}$, $T \in (0, R]$ and $u \in C_s^{2+\alpha}(\bar{Q}_{3R/2, T}(x^0))$ that satisfies*

$$\begin{cases} Lu = f & \text{on } Q_{3R/2, T}(x^0), \\ u(0, \cdot) = g & \text{on } \bar{B}_{3R/2}(x^0), \end{cases} \quad (3.58)$$

the following estimate holds

$$\|u\|_{C_s^{2+\alpha}(\bar{Q}_{R, T}(x^0))} \leq C \left(\|f\|_{C_s^\alpha(\bar{Q}_{3R/2, T}(x^0))} + \|g\|_{C_s^{2+\alpha}(\bar{B}_{3R/2}(x^0))} + \|u\|_{C(\bar{Q}_{3R/2, T}(x^0))} \right). \quad (3.59)$$

Proof. The proof is a blend of the localizing technique used in [24, Theorem 8.11.1] and the method of freezing the coefficients. Fix $R > 0$ and $T \in (0, R]$. Let $\varphi : \mathbb{R} \rightarrow [0, 1]$ be a smooth function such that $\varphi(t) = 0$ for $t < 0$, and $\varphi(t) = 1$ for $t > 1$. Let

$$R_n = R \sum_{k=0}^n \frac{1}{3^{-k}},$$

and consider the sequence of smooth cutoff functions $\{\varphi_n\}_{n \geq 1} \subset C^\infty(\bar{\mathbb{R}}^d)$ defined by

$$\varphi_n(x) := \varphi\left(\frac{R_{n+1} - |x|}{R_{n+1} - R_n}\right), \quad \forall x \in \bar{\mathbb{H}},$$

so that $0 \leq \varphi_n \leq 1$ and $\varphi_n|_{B_{R_n}} \equiv 1$ and $\varphi_n|_{B_{R_{n+1}}^c} \equiv 0$, where $B_{R_{n+1}}^c$ denotes the complement of $B_{R_{n+1}}$ in \mathbb{R}^d . Also, by direct calculation, we can find a positive constant c , independent of n and R , such that

$$\|\varphi_n\|_{C_s^\alpha(\bar{\mathbb{H}})}, \|(\varphi_n)_{x_i}\|_{C_s^\alpha(\bar{\mathbb{H}})}, \|x_d(\varphi_n)_{x_i x_j}\|_{C_s^\alpha(\bar{\mathbb{H}})}, \|(\varphi_n)_{x_i x_j}\|_{C_s^\alpha(\bar{\mathbb{H}})} \leq c3^{3n}R^{-3}. \quad (3.60)$$

We denote $r := 3^{-3} < 1$ and set

$$\alpha_n := \|u\varphi_n\|_{C_s^{2+\alpha}(\bar{\mathbb{H}}_T)}. \quad (3.61)$$

We denote by L_0 the operator with constant coefficients obtained by freezing the coefficients of L at $(0, x^0)$. Proposition A.1 shows there exists a positive constant C , depending only on K , δ and ν , such that

$$\alpha_n = \|u\varphi_n\|_{C_s^{2+\alpha}(\bar{\mathbb{H}}_T)} \leq C \left(\|L_0(u\varphi_n)\|_{C_s^\alpha(\bar{\mathbb{H}}_T)} + \|g\varphi_n\|_{C_s^{2+\alpha}(\bar{\mathbb{H}})} \right), \quad (3.62)$$

and so

$$\alpha_n \leq C \left(\|L(u\varphi_n)\|_{C_s^\alpha(\bar{\mathbb{H}}_T)} + \|(L - L_0)(u\varphi_n)\|_{C_s^\alpha(\bar{\mathbb{H}}_T)} + \|g\varphi_n\|_{C_s^{2+\alpha}(\bar{\mathbb{H}})} \right). \quad (3.63)$$

We have $L(u\varphi_n) = \varphi_n Lu - [L, \varphi_n]u$, where, by direct calculation,

$$[L, \varphi_n]u = \sum_{i,j=1}^d 2x_d a_{ij}(t, x) u_{x_i}(\varphi_n)_{x_j} + \sum_{i=1}^d b_i(t, x) u(\varphi_n)_{x_i} + \sum_{i,j=1}^d x_d a_{ij}(t, x) u(\varphi_n)_{x_i x_j}. \quad (3.64)$$

By the analogue of the [19, Inequality (4.7)] for standard Hölder norms, we have

$$\|\varphi_n Lu\|_{C_s^\alpha(\bar{\mathbb{H}}_T)} \leq c \|Lu\|_{C_s^\alpha(\bar{Q}_{R_{n+1}, T})} \|\varphi_n\|_{C_s^\alpha(\bar{\mathbb{H}})},$$

and by (3.60), there is a positive constant c such that

$$\|\varphi_n Lu\|_{C_s^\alpha(\bar{\mathbb{H}}_T)} \leq cr^{-n}R^{-3} \|f\|_{C_s^\alpha(\bar{Q}_{3R/2, T})}. \quad (3.65)$$

From properties (2.13) and (2.14) of the coefficients a_{ij} , b_i and c on $\bar{\mathbb{H}}_{2, T}$, we can find a positive constant C , depending only on K and d , such that

$$\|[L, \varphi_n]u\|_{C_s^\alpha(\bar{\mathbb{H}}_T)} \leq Cr^{-n}R^{-3} \left(\|x_d(u\varphi_{n+1})_{x_i}\|_{C_s^\alpha(\bar{\mathbb{H}}_T)} + \|u\varphi_{n+1}\|_{C_s^\alpha(\bar{\mathbb{H}}_T)} \right). \quad (3.66)$$

The interpolation inequality (3.8) in Lemma 3.2 gives us, for any $\varepsilon \in (0, 1)$,

$$\|x_d(u\varphi_{n+1})_{x_i}\|_{C_s^\alpha(\bar{\mathbb{H}}_T)} + \|u\varphi_{n+1}\|_{C_s^\alpha(\bar{\mathbb{H}}_T)} \leq \varepsilon \|u\varphi_{n+1}\|_{C_s^{2+\alpha}(\bar{\mathbb{H}}_T)} + C\varepsilon^{-m} \|u\varphi_{n+1}\|_{C(\bar{\mathbb{H}}_T)}. \quad (3.67)$$

Hence, the preceding inequality together with (3.65) and (3.66), give us

$$\begin{aligned} \|L(u\varphi_n)\|_{C_s^\alpha(\bar{\mathbb{H}}_T)} &\leq Cr^{-n}R^{-3} \left(\|f\|_{C_s^\alpha(\bar{Q}_{3R/2, T})} + \varepsilon \|u\varphi_{n+1}\|_{C_s^{2+\alpha}(\bar{\mathbb{H}}_T)} \right. \\ &\quad \left. + \varepsilon^{-m} \|u\varphi_{n+1}\|_{C(\bar{\mathbb{H}}_T)} \right). \end{aligned} \quad (3.68)$$

Next, we estimate the term $(L - L_0)(u\varphi_n)$ in (3.63), that is,

$$\begin{aligned} -(L - L_0)(u\varphi_n) &= \sum_{i,j=1}^d x_d (a_{ij}(t, x) - a_{ij}(0, x^0)) (u\varphi_n)_{x_i x_j} \\ &\quad + \sum_{i=1}^d (b_i(t, x) - b_i(0, x^0)) (u\varphi_n)_{x_i} + (c(t, x) - c(0, x^0)) (u\varphi_n). \end{aligned} \quad (3.69)$$

We have:

Claim 3.9. *There is a constant $C = C(K, R^*, d, \alpha)$ such that, for any $\varepsilon \in (0, 1)$, we have*

$$\begin{aligned} \|(L - L_0)(u\varphi_n)\|_{C_s^\alpha(\mathbb{H}_T)} &\leq C \left(R^{\alpha/2} + r^{-n} R^{-3} \varepsilon \right) \|u\varphi_{n+1}\|_{C_s^{2+\alpha}(\mathbb{H}_T)} \\ &\quad + C r^{-n} R^{-3} \varepsilon^{-m} \|u\varphi_{n+1}\|_{C(\mathbb{H}_T)}. \end{aligned} \quad (3.70)$$

where m is the constant appearing in Lemma 3.2.

Proof of Claim 3.9. From the Hölder continuity (2.14) and boundedness (2.13) of the coefficients a_{ij} on $\overline{\mathbb{H}}_{2,T}$, we can find a positive constant C , depending only on K and d , such that

$$\begin{aligned} &\|x_d (a_{ij}(t, x) - a_{ij}(0, x^0)) (u\varphi_n)_{x_i x_j}\|_{C_s^\alpha(\mathbb{H}_T)} \\ &\leq C R^{\alpha/2} \|x_d (u\varphi_n)_{x_i x_j}\|_{C_s^\alpha(\mathbb{H}_T)} + C \|x_d (u\varphi_n)_{x_i x_j}\|_{C(\mathbb{H}_T)}. \end{aligned} \quad (3.71)$$

Using the following calculation in the preceding inequality

$$\begin{aligned} \|x_d (u\varphi_n)_{x_i x_j}\|_{C_s^\alpha(\mathbb{H}_T)} &\leq \|x_d u_{x_i x_j} \varphi_n\|_{C_s^\alpha(\mathbb{H}_T)} + \|x_d u_{x_i} (\varphi_n)_{x_j}\|_{C_s^\alpha(\mathbb{H}_T)} + \|x_d u (\varphi_n)_{x_i x_j}\|_{C_s^\alpha(\mathbb{H}_T)} \\ &\leq [x_d (u\varphi_{n+1})_{x_i x_j}]_{C_s^\alpha(\mathbb{H}_T)} + c r^{-n} R^{-3} \left(\|x_d (u\varphi_{n+1})_{x_i x_j}\|_{C(\mathbb{H}_T)} \right. \\ &\quad \left. + \|x_d (u\varphi_{n+1})_{x_i}\|_{C_s^\alpha(\mathbb{H}_T)} + \|x_d u \varphi_{n+1}\|_{C_s^\alpha(\mathbb{H}_T)} \right), \end{aligned}$$

together with the interpolation inequality (3.9) in Lemma 3.2 applied to $u\varphi_{n+1}$,

$$\begin{aligned} &\|x_d (u\varphi_{n+1})_{x_i x_j}\|_{C(\mathbb{H}_T)} + \|x_d (u\varphi_{n+1})_{x_i}\|_{C_s^\alpha(\mathbb{H}_T)} + \|u\varphi_{n+1}\|_{C_s^\alpha(\mathbb{H}_T)} \\ &\leq \varepsilon \|u\varphi_{n+1}\|_{C_s^{2+\alpha}(\mathbb{H}_T)} + C \varepsilon^{-m} \|u\varphi_{n+1}\|_{C(\mathbb{H}_T)}, \end{aligned}$$

we obtain in (3.71)

$$\begin{aligned} &\|x_d (a_{ij}(t, x) - a_{ij}(0, x^0)) (u\varphi_n)_{x_i x_j}\|_{C_s^\alpha(\mathbb{H}_T)} \\ &\leq C R^{\alpha/2} [x_d (u\varphi_{n+1})]_{C_s^\alpha(\mathbb{H}_T)} + C r^{-n} R^{-3} \varepsilon \|u\varphi_{n+1}\|_{C_s^{2+\alpha}(\mathbb{H}_T)} + C r^{-n} R^{-3} \varepsilon^{-m} \|u\varphi_{n+1}\|_{C(\mathbb{H}_T)} \\ &\leq C \left(R^{\alpha/2} + r^{-n} R^{-3} \varepsilon \right) \|u\varphi_{n+1}\|_{C_s^{2+\alpha}(\mathbb{H}_T)} + C r^{-n} R^{-3} \varepsilon^{-m} \|u\varphi_{n+1}\|_{C(\mathbb{H}_T)}. \end{aligned}$$

A similar argument gives us

$$\begin{aligned} &\|(b_i(t, x) - b_i(0, x^0)) (u\varphi_n)_{x_i}\|_{C_s^\alpha(\mathbb{H}_T)} + \|(c(t, x) - c(0, x^0)) (u\varphi_n)\|_{C_s^\alpha(\mathbb{H}_T)} \\ &\leq C r^{-n} R^{-3} \varepsilon \|u\varphi_{n+1}\|_{C_s^{2+\alpha}(\mathbb{H}_T)} + C r^{-n} R^{-3} \varepsilon^{-m} \|u\varphi_{n+1}\|_{C(\mathbb{H}_T)}, \end{aligned}$$

and so, using the preceding inequalities in (3.69), we obtain the estimate (3.70). \square

Combining (3.68), (3.70) and (3.63), we obtain

$$\begin{aligned} \alpha_n &\leq Cr^{-n}R^{-3} \left(\|f\|_{C_s^\alpha(\bar{Q}_{3R/2,T})} + \|g\|_{C_s^{2+\alpha}(\bar{B}_{3R/2})} \right) \\ &\quad + C \left(R^{\alpha/2} + r^{-n}R^{-3}\varepsilon \right) \alpha_{n+1} + Cr^{-n}R^{-3}\varepsilon^{-m} \|u\|_{C(\bar{Q}_{3R/2,T})}. \end{aligned} \quad (3.72)$$

We multiply the inequality (3.72) by δ^n , where $\delta > 0$ is chosen such that

$$r^{-(m+1)}\delta \leq 1/2. \quad (3.73)$$

Next, we choose $R^* > 0$ such that $CR^{*\alpha/2} = \delta/2$. For $R \in (0, R^*]$, we choose $\varepsilon = \varepsilon(n, R) \in (0, 1)$ such that $Cr^{-n}R^{-3}\varepsilon = \delta/2$. With this choice of δ , R^* and ε , inequality (3.72) yields, for all $0 < R \leq R^*$,

$$\begin{aligned} \delta^n \alpha_n &\leq CR^{-3}(r^{-1}\delta)^n \left(\|f\|_{C_s^\alpha(\bar{Q}_{3R/2,T})} + \|g\|_{C_s^{2+\alpha}(\bar{B}_{3R/2})} \right) \\ &\quad + \delta^{n+1} \alpha_{n+1} + (2C)^{m+1} R^{-3(m+1)} \delta^{-m} (r^{-(m+1)}\delta)^n \|u\|_{C(\bar{B}_{3R/2,T})}. \end{aligned}$$

By (3.73), we also have $r^{-1}\delta \leq 1/2$. Then, by choosing

$$C_1 := \max \left\{ CR^{-3}, (2C)^{m+1} R^{-3(m+1)} \delta^{-m} \right\},$$

we obtain

$$\delta^n \alpha_n \leq C_1 \frac{1}{2^n} \left(\|f\|_{C_s^\alpha(\bar{Q}_{3R/2,T})} + \|g\|_{C_s^{2+\alpha}(\bar{B}_{3R/2})} \right) + \delta^{n+1} \alpha_{n+1} + C_1 \frac{1}{2^n} \|u\|_{C(\bar{Q}_{3R/2,T})}. \quad (3.74)$$

Summing inequality (3.74) yields

$$\begin{aligned} \sum_{n=0}^{\infty} \delta^n \alpha_n &\leq C_1 \left(\|f\|_{C_s^\alpha(\bar{Q}_{3R/2,T})} + \|g\|_{C_s^{2+\alpha}(\bar{B}_{3R/2})} \right) \sum_{n=0}^{\infty} \frac{1}{2^n} \\ &\quad + \sum_{n=0}^{\infty} \delta^{n+1} \alpha_{n+1} + C_1 \|u\|_{C(\bar{Q}_{3R/2,T})} \sum_{n=0}^{\infty} \frac{1}{2^n}. \end{aligned}$$

The sum $\sum_{n=0}^{\infty} \delta^n \alpha_n$ is well-defined because we assumed $u \in C_s^{2+\alpha}(\bar{Q}_{3R/2,T})$, for all $R \in (0, R^*]$ and $T \in (0, R]$, while $\delta \in (0, 1)$. By subtracting the term $\sum_{n=1}^{\infty} \delta^n \alpha_n$ from both sides of the preceding inequality, we obtain the desired inequality (3.59). \square

3.4. Local a priori interior estimates. In order to establish the local interior estimates, we need to track the dependency of the constant N appearing in [24, Lemma 9.2.1 & Theorem 9.2.2] on the constant of uniform ellipticity and on the supremum and Hölder norms of the coefficients. Lemma 3.11 and Proposition 3.12 apply to a parabolic operator

$$-\bar{L}u := -u_t + \sum_{i,j=1}^d \bar{a}_{ij} u_{x_i x_j} + \sum_{i=1}^d \bar{b}_i u_{x_i} + \bar{c}u, \quad (3.75)$$

whose coefficients obey

Hypothesis 3.10. There are positive constants δ_1 , K_1 and λ_1 such that

- (1) $(\bar{a}_{ij}(t, x))$ is a symmetric, positive definite matrix, for all $t \in [0, T]$ and $x \in \mathbb{R}^d$.
- (2) The diffusion matrix \bar{a} is non-degenerate

$$\sum_{i,j=1}^d \bar{a}_{ij}(t, x) \xi_i \xi_j \geq \delta_1 |\xi|^2, \quad \forall \xi \in \mathbb{R}^d, t \in [0, T], x \in \mathbb{R}^d. \quad (3.76)$$

(3) The coefficients \bar{a}_{ij} , \bar{b}_i and \bar{c} are uniformly Hölder continuous on $[0, T] \times \mathbb{R}^d$,

$$\|\bar{a}_{ij}\|_{C_\rho^\alpha([0, T] \times \mathbb{R}^d)} + \|\bar{b}_i\|_{C_\rho^\alpha([0, T] \times \mathbb{R}^d)} + \|\bar{c}\|_{C_\rho^\alpha([0, T] \times \mathbb{R}^d)} \leq K_1. \quad (3.77)$$

(4) The zeroth order coefficient, \bar{c} , is bounded from above,

$$\bar{c}(t, x) \leq \lambda_1 \quad \forall t \in [0, T], x \in \mathbb{R}^d. \quad (3.78)$$

The difference between the statements of Lemma 3.11 and [24, Lemmas 9.2.1 & 8.9.1] is that we explicitly give the dependency of the constant N_2 on δ_1 and K_1 ; the proofs are the same except that at each step we explicitly determine the dependency of the constants appearing in the estimate (3.82) on δ_1 and K_1 .

Lemma 3.11 (A priori estimate for a simple parabolic operator with constant coefficients). *Assume that (\bar{a}_{ij}) in (3.75) is a constant matrix obeying (3.76), $\bar{b}_i = 0$, and $\bar{c} = 0$. Then there are positive constants,*

$$N_1 = N_1(d, \alpha, T), \quad (3.79)$$

$$N_2 = N_1 \max\{1, \delta_1^{-1}\} \max\{1, K_1\} (1 + \delta_1^{-\alpha/2}) (1 + K_1^{\alpha/2}), \quad (3.80)$$

such that, for any solution $u \in C_\rho^{2+\alpha}([0, T] \times \mathbb{R}^d)$ to

$$\begin{cases} \bar{L}u = f & \text{on } (0, T) \times \mathbb{R}^d, \\ u(0, \cdot) = g & \text{on } \mathbb{R}^d, \end{cases} \quad (3.81)$$

with $f \in C_\rho^\alpha([0, T] \times \mathbb{R}^d)$ and $g \in C_\rho^{2+\alpha}(\mathbb{R}^d)$, we have

$$\|u\|_{C_\rho^{2+\alpha}([0, T] \times \mathbb{R}^d)} \leq N_2 \left(\|f\|_{C_\rho^\alpha([0, T] \times \mathbb{R}^d)} + \|g\|_{C_\rho^{2+\alpha}(\mathbb{R}^d)} \right). \quad (3.82)$$

The proof of Lemma 3.11 can be found in Appendix B. The statement of Proposition 3.12 is the same as that of [24, Theorems 9.2.2 & 8.9.2] except that in the estimate (3.86), the dependency of the constant N_4 on δ_1 and K_1 is made explicit in (3.83), (3.84) and (3.85).

Proposition 3.12 (A priori estimate for a parabolic operator with variable coefficients). *Assume Hypothesis 3.10. Then there are positive constants*

$$p = p(\alpha) \geq 1, \quad (3.83)$$

$$N_3 = N_3(d, \alpha, T), \quad (3.84)$$

$$N_4 = N_3 e^{\lambda_1 T} \left(1 + \delta_1^{-p} + K_1^p \right), \quad (3.85)$$

such that, for any solution $u \in C_\rho^{2+\alpha}([0, T] \times \mathbb{R}^d)$ to

$$\begin{cases} \bar{L}u = f & \text{on } (0, T) \times \mathbb{R}^d, \\ u(0, \cdot) = g & \text{on } \mathbb{R}^d, \end{cases}$$

we have

$$\|u\|_{C_\rho^{2+\alpha}([0, T] \times \mathbb{R}^d)} \leq N_4 \left(\|f\|_{C_\rho^\alpha([0, T] \times \mathbb{R}^d)} + \|g\|_{C_\rho^{2+\alpha}(\mathbb{R}^d)} \right). \quad (3.86)$$

The proof of Proposition 3.12 can be found in Appendix B. Next, we have the

Proposition 3.13 (Local estimates for parabolic operators with variable coefficients). *Assume Hypothesis 3.10 and that $R > 0$. Then there are positive constants*

$$p = p(\alpha) \geq 1, \quad (3.87)$$

$$N_3 = N_3(d, \alpha, T, R), \quad (3.88)$$

$$N_4 = N_3 e^{\lambda_1 T} \left(1 + \delta_1^{-p} + K_1^p\right), \quad (3.89)$$

such that for any $x^0 \in \mathbb{R}^d$ and any solution $u \in C_\rho^{2+\alpha}(\bar{Q}_{2R,T}(x^0))$ to

$$\begin{cases} \bar{L}u = f & \text{on } Q_{2R,T}(x^0), \\ u(0, \cdot) = g & \text{on } \bar{B}_{2R}(x^0), \end{cases}$$

we have

$$\begin{aligned} \|u\|_{C_\rho^{2+\alpha}(\bar{Q}_{R,T}(x^0))} &\leq N_4 \left(\|f\|_{C_\rho^\alpha(\bar{Q}_{2R,T}(x^0))} + \|g\|_{C_\rho^{2+\alpha}(\bar{B}_{2R}(x^0))} \right. \\ &\quad \left. + \|u\|_{C(\bar{Q}_{2R,T}(x^0))} \right). \end{aligned} \quad (3.90)$$

Proof. The proof follows by the same argument as in Theorem 3.8 with the following modifications:

- In inequality (3.62), instead of applying Proposition A.1, we apply Proposition 3.12.
- We use the interpolation inequalities for classical Hölder spaces $C_\rho^{2+\alpha}$ ([24, Theorem 8.8.1]), instead of the interpolation inequalities suitable for the Hölder spaces $C_s^{2+\alpha}$ (Lemma 3.2).

This completes the proof. \square

We now consider estimates for the operator L in (1.2).

Proposition 3.14 (Interior local estimates). *There is a positive constant $p = p(\alpha)$, and for any $0 < R \leq R^*$, with R^* as in Theorem 3.8, there is a positive constant $C = C(d, \alpha, T, K, \delta, R^*, R)$, such that for any $x^0 \in \mathbb{H}$ satisfying $x_d^0 - 2R \geq R^*/2$, and for any solution $u \in C_\rho^{2+\alpha}(\bar{Q}_{2R,T}(x^0))$ to the inhomogeneous initial value problem*

$$\begin{cases} Lu = f & \text{on } Q_{2R,T}(x^0), \\ u(0, \cdot) = g & \text{on } \bar{B}_{2R}(x^0), \end{cases}$$

we have

$$\begin{aligned} \|u\|_{C_\rho^{2+\alpha}(\bar{Q}_{R,T}(x^0))} &\leq C \left(\|f\|_{\mathcal{C}_p^\alpha(\bar{Q}_{2R,T}(x^0))} + \|g\|_{\mathcal{C}_p^{2+\alpha}(\bar{B}_{2R}(x^0))} \right. \\ &\quad \left. + \|u\|_{\mathcal{C}_p^0(\bar{Q}_{2R,T}(x^0))} \right). \end{aligned} \quad (3.91)$$

Proof. From Proposition 3.13, the linear growth estimate (2.17), and the fact that the matrix $(x_d a_{ij}(t, x))$ is uniformly elliptic on $\bar{\mathbb{H}}_T \setminus \mathbb{H}_{R^*/2,T}$ by (2.12) and (2.15), we obtain

$$\begin{aligned} \|u\|_{C_\rho^{2+\alpha}(\bar{Q}_{R,T}(x^0))} &\leq C_1 (1 + |x^0|)^p \left(\|f\|_{C_\rho^\alpha(\bar{Q}_{2R,T}(x^0))} + \|g\|_{C_\rho^{2+\alpha}(\bar{B}_{2R}(x^0))} \right. \\ &\quad \left. + \|u\|_{C(\bar{Q}_{2R,T}(x^0))} \right), \end{aligned} \quad (3.92)$$

where C_1 is a positive constant depending only on T, K, δ, R^* and R .

Claim 3.15. *Given a function $v \in C_\rho^{2+\alpha}(\bar{Q}_{2R,T}(x^0))$, there is a positive constant C_2 , depending only in R^* , p and α , such that for all $R \in (0, R^*]$ and $x^0 \in \mathbb{H}_T$, we have*

$$(1 + |x^0|)^p \|v\|_{C_\rho^\alpha(\bar{Q}_{2R,T}(x^0))} \leq C_2 \|v\|_{\mathcal{C}_p^\alpha(\bar{Q}_{2R,T}(x^0))}. \quad (3.93)$$

Proof of Claim 3.15. Recall that, by definition (2.8),

$$\|(1 + |x|)^p v\|_{C_\rho^\alpha(\bar{Q}_{2R,T}(x^0))} = \|v\|_{\mathcal{C}_p^\alpha(\bar{Q}_{2R,T}(x^0))}.$$

We may write

$$(1 + |x^0|)^p |v(t, x)| = \left(\frac{1 + |x^0|}{1 + |x|} \right)^p (1 + |x|)^p |v(t, x)|, \quad \forall (t, x) \in \bar{Q}_{2R,T}(x^0).$$

We can find a constant $C_2 = C_2(R^*, p)$ such that

$$\left(\frac{1 + |x^0|}{1 + |x|} \right)^p \leq C_2, \quad \forall x \in \bar{B}_{2R}(x^0), \quad \forall 0 < R < R^*, \quad (3.94)$$

which implies

$$(1 + |x^0|)^p \|v\|_{C(\bar{Q}_{2R,T}(x^0))} \leq C_2 \|(1 + |x|)^p v\|_{C(\bar{Q}_{2R,T}(x^0))}. \quad (3.95)$$

Next, we have

$$\begin{aligned} (1 + |x^0|)^p [v]_{C_\rho^\alpha(\bar{Q}_{2R,T}(x^0))} &= (1 + |x^0|)^p \left[\frac{1}{(1 + |x|)^p} (1 + |x|)^p v \right]_{C_\rho^\alpha(\bar{Q}_{2R,T}(x^0))} \\ &\leq (1 + |x^0|)^p \left[\frac{1}{(1 + |x|)^p} \right]_{C_\rho^\alpha(\bar{B}_{2R}(x^0))} \|(1 + |x|)^p v\|_{C(\bar{Q}_{2R,T}(x^0))} \\ &\quad + (1 + |x^0|)^p \left\| \frac{1}{(1 + |x|)^p} \right\|_{C(\bar{B}_{2R}(x^0))} [(1 + |x|)^p v]_{C_\rho^\alpha(\bar{Q}_{2R,T}(x^0))}. \end{aligned}$$

As in (3.94), there is a (possibly larger) constant $C_2 = C_2(R^*, p, \alpha)$ such that

$$(1 + |x^0|)^p \left[\frac{1}{(1 + |x|)^p} \right]_{C_\rho^\alpha(\bar{B}_{2R}(x^0))} \leq C_2.$$

Therefore, we obtain

$$(1 + |x^0|)^p [v]_{C_\rho^\alpha(\bar{Q}_{2R,T}(x^0))} \leq C_2 \|(1 + |x|)^p v\|_{C(\bar{Q}_{2R,T}(x^0))} + C_2 [(1 + |x|)^p v]_{C_\rho^\alpha(\bar{Q}_{2R,T}(x^0))}. \quad (3.96)$$

Combining inequalities (3.95) and (3.96) yields the desired inequality (3.93). \square

Claim 3.15 implies that

$$\begin{aligned} (1 + |x^0|)^p \|f\|_{C_\rho^\alpha(\bar{Q}_{2R,T}(x^0))} &\leq C_2 \|f\|_{\mathcal{C}_p^\alpha(\bar{Q}_{2R,T}(x^0))}, \\ (1 + |x^0|)^p \|g\|_{C_\rho^{2+\alpha}(\bar{B}_{2R}(x^0))} &\leq C_2 \|g\|_{\mathcal{C}_p^{2+\alpha}(\bar{B}_{2R}(x^0))}, \\ (1 + |x^0|)^p \|u\|_{C(\bar{Q}_{2R,T}(x^0))} &\leq C_2 \|u\|_{\mathcal{C}_p^0(\bar{Q}_{2R,T}(x^0))}, \end{aligned}$$

and so, the interior local estimate (3.91) follows from the preceding inequalities and (3.92). \square

3.5. Global a priori estimates and existence of solutions. The goal of this subsection is to establish Theorem 1.1. For this purpose, we need to first prove the analogue of Theorem 1.1 when the coefficients are uniformly Hölder continuous on $\mathbb{H}_T \setminus \mathbb{H}_{2,T} = (0, T) \times \mathbb{R}^{d-1} \times [2, \infty)$.

Hypothesis 3.16. In addition to the conditions in Assumption 2.2, assume that there is a positive constant K_2 such that the coefficients of L obey

$$\|x_d a_{ij}\|_{C_\rho^\alpha(\overline{\mathbb{H}_T \setminus \mathbb{H}_{2,T}})} + \|b_i\|_{C_\rho^\alpha(\overline{\mathbb{H}_T \setminus \mathbb{H}_{2,T}})} + \|c\|_{C_\rho^\alpha(\overline{\mathbb{H}_T \setminus \mathbb{H}_{2,T}})} \leq K_2. \quad (3.97)$$

We first derive global a priori estimates of solutions in the case of bounded coefficients.

Lemma 3.17 (Global estimates in the case of parabolic operators with bounded coefficients). *Suppose Hypothesis 3.16 is satisfied. There exists a positive constant $C = C(T, \alpha, d, K_2, \delta, \nu)$ such that for any solution $u \in \mathcal{C}_{\text{loc}}^{2+\alpha}(\overline{\mathbb{H}_T})$ to (1.1), such that $Lu \in \mathcal{C}^\alpha(\overline{\mathbb{H}_T})$ and $u(0, \cdot) \in \mathcal{C}^{2+\alpha}(\overline{\mathbb{H}})$, we have $u \in \mathcal{C}^{2+\alpha}(\overline{\mathbb{H}_T})$ and satisfies the global estimate*

$$\|u\|_{\mathcal{C}^{2+\alpha}(\overline{\mathbb{H}_T})} \leq C \left(\|Lu\|_{\mathcal{C}^\alpha(\overline{\mathbb{H}_T})} + \|u(0, \cdot)\|_{\mathcal{C}^{2+\alpha}(\overline{\mathbb{H}})} \right). \quad (3.98)$$

Proof. We shall apply a covering argument with the aid of Theorem 3.8 and Proposition 3.13. It is enough to prove the statement for $T > 0$ small. Let $R^* > 0$ be defined as in Theorem 3.8 and choose $T \in (0, R^*]$. Let $\{z^k : k \geq 1\}$ be a sequence of points in $\partial\mathbb{H}$ such that

$$\mathbb{H}_{R^*/2, T} \subset \bigcup_{k \geq 1} Q_{R^*, T}(z^k), \quad (3.99)$$

and let $\{w^l : l \geq 1\}$ be a sequence of points in $\mathbb{H}_T \setminus \mathbb{H}_{R^*/2, T}$ such that

$$\mathbb{H}_T \setminus \mathbb{H}_{R^*/2, T} \subset \bigcup_{l \geq 1} Q_{R^*/8, T}(w^l), \quad (3.100)$$

and assume

$$Q_{R^*/4, T}(w^l) \cap \mathbb{H}_{R^*/4, T} = \emptyset, \quad \forall l \geq 1. \quad (3.101)$$

We apply the a priori boundary estimate (3.59) to u with $R = R^*$, $f = Lu$ and $g = u(0, \cdot)$ on $Q_{R^*, T}(z^k)$. Then, we can find a positive constant C_1 , depending at most on R^* , K_2 , δ , ν , such that

$$\begin{aligned} \|u\|_{C_s^{2+\alpha}(\bar{Q}_{R^*, T}(z^k))} &\leq C_1 \left(\|Lu\|_{C_s^\alpha(\bar{Q}_{3R^*/2, T}(z^k))} + \|u(0, \cdot)\|_{C_s^{2+\alpha}(\bar{Q}_{3R^*/2, T}(z^k))} \right. \\ &\quad \left. + \|u\|_{C(\bar{Q}_{3R^*/2, T}(z^k))} \right). \end{aligned}$$

Using definitions (2.8) of $\mathcal{C}^\alpha(\overline{\mathbb{H}_T})$, and (2.9) of $\mathcal{C}^{2+\alpha}(\overline{\mathbb{H}})$, with $q = 0$, Remark 2.1 and the hypotheses that $Lu \in \mathcal{C}^\alpha(\overline{\mathbb{H}_T})$ and $u(0, \cdot) \in \mathcal{C}^{2+\alpha}(\overline{\mathbb{H}})$, we obtain

$$\|u\|_{C_s^{2+\alpha}(\bar{Q}_{R^*, T}(z^k))} \leq C_1 \left(\|Lu\|_{\mathcal{C}^\alpha(\overline{\mathbb{H}_T})} + \|u(0, \cdot)\|_{\mathcal{C}^{2+\alpha}(\overline{\mathbb{H}})} + \|u\|_{C(\overline{\mathbb{H}_T})} \right),$$

and inequality (3.53) ensures

$$\|u\|_{C_s^{2+\alpha}(\bar{Q}_{R^*, T}(z^k))} \leq C_1 \left(\|Lu\|_{\mathcal{C}^\alpha(\overline{\mathbb{H}_T})} + \|u(0, \cdot)\|_{\mathcal{C}^{2+\alpha}(\overline{\mathbb{H}})} \right), \quad \forall k \geq 1. \quad (3.102)$$

From our Hypothesis 3.16, the coefficients $x_d a_{ij}$, b_i and c are in $C_\rho^\alpha(\overline{\mathbb{H}_T \setminus \mathbb{H}_{2,T}})$. By Assumption 2.2, we have that $x_d a_{ij}$, b_i and c are in $C_s^\alpha(\overline{\mathbb{H}_{2,T} \setminus \mathbb{H}_{R^*/4, T}})$. Since the metrics s and ρ are equivalent on $\mathbb{R} \times [R^*/4, 2]$, by Remark 2.1, there is a positive constant K_1 , depending on K_2 and R^* , such that

$$\|x_d a_{ij}\|_{C_\rho^\alpha(\overline{\mathbb{H}_T \setminus \mathbb{H}_{R^*/4, T}})} + \|b_i\|_{C_\rho^\alpha(\overline{\mathbb{H}_T \setminus \mathbb{H}_{R^*/4, T}})} + \|c\|_{C_\rho^\alpha(\overline{\mathbb{H}_T \setminus \mathbb{H}_{R^*/4, T}})} \leq K_1,$$

and so the conditions of Hypothesis 3.10 are obeyed on $\overline{\mathbb{H}_T \setminus \mathbb{H}_{R^*/4,T}}$. This is enough to ensure we may apply Proposition 3.13 to u with $f = Lu$ and $g = u(0, \cdot)$ on $Q_{R^*/8,T}(w^l)$ and so there is a positive constant C_2 , depending at most on R^* , K_1 , δ , ν , giving

$$\begin{aligned} \|u\|_{C_\rho^{2+\alpha}(\bar{Q}_{R^*/8,T}(w^l))} &\leq C_2 \left(\|Lu\|_{C_\rho^\alpha(\bar{Q}_{R^*/4,T}(w^l))} + \|u(0, \cdot)\|_{C_\rho^{2+\alpha}(\bar{Q}_{R^*/4,T}(w^l))} \right. \\ &\quad \left. + \|u\|_{C(\bar{Q}_{R^*/4,T}(w^l))} \right), \quad \forall l \geq 1. \end{aligned}$$

By (3.101) and Remark 2.1, we obtain

$$\begin{aligned} \|u\|_{C_\rho^{2+\alpha}(\bar{Q}_{R^*/8,T}(w^l))} &\leq C_2 \left(\|Lu\|_{\mathcal{C}^\alpha(\bar{\mathbb{H}})} + \|u(0, \cdot)\|_{\mathcal{C}^{2+\alpha}(\bar{\mathbb{H}}_T)} \right. \\ &\quad \left. + \|u\|_{C(\bar{Q}_{R^*/4,T}(w^l))} \right), \quad \forall l \geq 1, \end{aligned}$$

and, by inequality (3.53) applied to $\|u\|_{C(\bar{Q}_{R^*/4,T}(w^l))}$, it follows

$$\|u\|_{C_\rho^{2+\alpha}(\bar{Q}_{R^*/8,T}(w^l))} \leq C_2 \left(\|Lu\|_{\mathcal{C}^\alpha(\bar{\mathbb{H}})} + \|u(0, \cdot)\|_{\mathcal{C}^{2+\alpha}(\bar{\mathbb{H}}_T)} \right), \quad \forall l \geq 1. \quad (3.103)$$

Combining inequalities (3.102) and (3.103) and making use of the inclusions (3.99) and (3.100), we obtain the global estimate (3.98). \square

Next, we establish the a priori global estimates in the case of coefficients with at most linear growth.

Lemma 3.18 (Global estimates for coefficients with linear growth). *There exists a positive constant $C = C(T, \alpha, d, K, \delta, \nu)$ such that for any solution $u \in \mathcal{C}_{\text{loc}}^{2+\alpha}(\bar{\mathbb{H}}_T)$ to (1.1), such that $Lu \in \mathcal{C}_p^\alpha(\bar{\mathbb{H}}_T)$ and $u(0, \cdot) \in \mathcal{C}_p^{2+\alpha}(\bar{\mathbb{H}})$, we have*

$$\|u\|_{\mathcal{C}^{2+\alpha}(\bar{\mathbb{H}}_T)} \leq C \left(\|Lu\|_{\mathcal{C}_p^\alpha(\bar{\mathbb{H}}_T)} + \|u(0, \cdot)\|_{\mathcal{C}_p^{2+\alpha}(\bar{\mathbb{H}})} \right), \quad (3.104)$$

where $p = p(\alpha)$ is the constant appearing in Proposition 3.14.

Proof. We shall apply a covering argument with the aid of Theorem 3.8 and Proposition 3.14. As in the proof of Lemma 3.17, we may assume without loss of generality that $0 < T \leq R^*$, where $R^* > 0$ is defined as in Theorem 3.8. Let z^k and w^l be the sequences of points considered in the proof of Lemma 3.17. Then, by applying Theorem 3.8 to u with $f = Lu$ and $g = u(0, \cdot)$ on $\bar{Q}_{R^*,T}(z^k)$, we obtain, for all $k \geq 1$,

$$\begin{aligned} \|u\|_{C_s^{2+\alpha}(\bar{Q}_{R^*,T}(z^k))} &\leq C \left(\|Lu\|_{C_s^\alpha(\bar{Q}_{3R^*/2,T}(z^k))} + \|u(0, \cdot)\|_{C_s^{2+\alpha}(\bar{B}_{3R^*/2}(z^k))} \right. \\ &\quad \left. + \|u\|_{C(\bar{Q}_{3R^*/2,T}(z^k))} \right). \end{aligned}$$

We notice that

$$\begin{aligned} \|Lu\|_{C_s^\alpha(\bar{Q}_{3R^*/2,T}(z^k))} &\leq C_1 \|(1 + |x|)^p Lu\|_{C_s^\alpha(\bar{Q}_{3R^*/2,T}(z^k))} \\ &= C_1 \|Lu\|_{\mathcal{C}_p^\alpha(\bar{Q}_{3R^*/2,T}(z^k))}, \\ \|u(0, \cdot)\|_{C_s^{2+\alpha}(\bar{B}_{3R^*/2}(z^k))} &\leq C_1 \|(1 + |x|)^p u(0, \cdot)\|_{C_s^{2+\alpha}(\bar{B}_{3R^*/2}(z^k))} \\ &= C_1 \|u(0, \cdot)\|_{\mathcal{C}_p^{2+\alpha}(\bar{B}_{3R^*/2}(z^k))}, \\ \|u\|_{C(\bar{Q}_{3R^*/2,T}(z^k))} &\leq C_1 \|(1 + |x|)^p u\|_{C(\bar{Q}_{3R^*/2,T}(z^k))} \\ &= C_1 \|u\|_{\mathcal{C}_p^0(\bar{Q}_{3R^*/2,T}(z^k))}, \end{aligned}$$

where the positive constant C_1 depends on R^* and p , but not on z^k . Therefore, we obtain, for all $k \geq 1$,

$$\|u\|_{C_s^{2+\alpha}(\bar{Q}_{R^*,T}(z^k))} \leq C_2 \left(\|Lu\|_{\mathcal{C}_p^\alpha(\bar{\mathbb{H}}_T)} + \|u(0, \cdot)\|_{\mathcal{C}_p^{2+\alpha}(\bar{\mathbb{H}})} + \|u\|_{\mathcal{C}_p^0(\bar{\mathbb{H}}_T)} \right),$$

for a positive constant C_2 depending at most on R^* , K , δ , ν , α , d . Because the collection of balls $\{Q_{R^*,T}(z^k) : k \geq 1\}$ covers $\mathbb{H}_{R^*/2,T}$ and as we may apply (3.54) to u with $f = Lu$ and $g = u(0, \cdot)$ with $q = p$, there is a positive constant C_3 , satisfying the same dependency on constants as C_2 , such that

$$\|u\|_{\mathcal{C}^{2+\alpha}(\bar{\mathbb{H}}_{R^*/2,T})} \leq C_3 \left(\|Lu\|_{\mathcal{C}_p^\alpha(\bar{\mathbb{H}}_T)} + \|u(0, \cdot)\|_{\mathcal{C}_p^{2+\alpha}(\bar{\mathbb{H}})} \right). \quad (3.105)$$

By applying Proposition 3.14 to u with $f = Lu$ and $g = u(0, \cdot)$ on $\bar{Q}_{R^*/8,T}(w^l)$, we obtain, for all $l \geq 1$,

$$\begin{aligned} \|u\|_{C_\rho^{2+\alpha}(\bar{Q}_{R^*/8,T}(w^l))} &\leq C_4 \left(\|Lu\|_{\mathcal{C}_p^\alpha(\bar{Q}_{R^*/4,T}(w^l))} + \|u(0, \cdot)\|_{\mathcal{C}_p^{2+\alpha}(\bar{B}_{R^*/4}(w^l))} \right. \\ &\quad \left. + \|u\|_{\mathcal{C}_p^0(\bar{Q}_{R^*/4,T}(w^l))} \right). \end{aligned} \quad (3.106)$$

Because the collection of balls $\{Q_{R^*/8,T}(w^l) : l \geq 1\}$ covers $\mathbb{H}_T \setminus \mathbb{H}_{R^*/2,T}$ and we may apply (3.54) to u with $f = Lu$ and $g = u(0, \cdot)$ with $q = p$, we obtain

$$\|u\|_{\mathcal{C}^{2+\alpha}(\overline{\mathbb{H}_T \setminus \mathbb{H}_{R^*/2,T}})} \leq C_5 \left(\|Lu\|_{\mathcal{C}_p^\alpha(\bar{\mathbb{H}}_T)} + \|u(0, \cdot)\|_{\mathcal{C}_p^{2+\alpha}(\bar{\mathbb{H}})} \right). \quad (3.107)$$

By combining inequalities (3.105) and (3.107), we obtain the desired estimate (3.104). \square

Next, we prove Theorem 1.1 in the case of bounded coefficients.

Proposition 3.19 (Existence and uniqueness for bounded coefficients). *Suppose Hypothesis 3.16 is satisfied. Let $f \in \mathcal{C}^\alpha(\bar{\mathbb{H}}_T)$ and $g \in \mathcal{C}^{2+\alpha}(\bar{\mathbb{H}})$. Then there exists a unique solution $u \in \mathcal{C}^{2+\alpha}(\bar{\mathbb{H}}_T)$ to (1.1) and u satisfies estimate (3.98).*

Proof. The proof employs the method used in proving existence of solutions to parabolic partial differential equations outlined in [24, §10.2] or [3, Theorem II.1.1]. We let $\hat{\mathcal{C}}^{2+\alpha}(\bar{\mathbb{H}}_T)$ denote the Banach space of functions $u \in \mathcal{C}^{2+\alpha}(\bar{\mathbb{H}}_T)$ such that $u(0, x) = 0$, for all $x \in \bar{\mathbb{H}}$. The spaces $\hat{C}_s^{2+\alpha}(\bar{\mathbb{H}}_T)$ and $\hat{C}_\rho^{2+\alpha}([0, T] \times \mathbb{R}^d)$ are defined similarly. Without loss of generality, we may assume $g = 0$ because $Lg \in \mathcal{C}^\alpha(\bar{\mathbb{H}}_T)$, when Hypothesis 3.16 holds, and so

$$L : \hat{\mathcal{C}}^{2+\alpha}(\bar{\mathbb{H}}_T) \rightarrow \mathcal{C}^\alpha(\bar{\mathbb{H}}_T)$$

is a well-defined operator. Our goal is to show that L is invertible and we accomplish this by constructing a bounded linear operator $M : \mathcal{C}^\alpha(\bar{\mathbb{H}}_T) \rightarrow \hat{\mathcal{C}}^{2+\alpha}(\bar{\mathbb{H}}_T)$ such that

$$\|LM - I_{\mathcal{C}^\alpha(\bar{\mathbb{H}}_T)}\| < 1. \quad (3.108)$$

For this purpose, we fix $r > 0$ and choose a sequence of points $\{x^n : n = 1, 2, \dots\}$ such that the collection of balls $\{B_r(x^n) : n = 1, 2, \dots\}$ covers the strip $\{x = (x', x_d) \in \mathbb{H} : 0 < x_d < r/2\}$. We may assume without loss of generality, there exists a positive constant N , depending only on the dimension d , such that at most N balls of the covering have non-empty intersection. Let $\{\varphi_n : n = 0, 1, \dots\}$ be a partition of unity subordinate to the open cover

$$(\mathbb{H} \setminus \{0 < x_d \leq r/4\}) \cup \bigcup_{n=1}^{\infty} B_r(x^n) = \mathbb{H},$$

such that

$$\text{supp } \varphi_0 \subset \mathbb{H} \setminus \{0 < x_d < r/4\} \text{ and } \text{supp } \varphi_n \subset \bar{B}_r(x^n), \quad \forall n \geq 1.$$

Without loss of generality, we may choose $\{\varphi_n\}_{n \geq 0}$ such that there is a positive constant c , independent of r and n , such that

$$\|\varphi_n\|_{C_\rho^{2+\alpha}(\mathbb{R}^d)} \leq cr^{-3}, \quad \forall n \geq 0. \quad (3.109)$$

We choose a sequence of non-negative, smooth cutoff functions, $\{\psi_n\}_{n \geq 0} \subset C^\infty(\bar{\mathbb{H}})$ such that $0 \leq \psi_n \leq 1$ on \mathbb{H} , for all $n \geq 0$, and

$$\psi_0(x) = \begin{cases} 0, & \text{for } 0 < x_d < r/8, \\ 1, & \text{for } x_d > r/4, \end{cases}$$

while for all $n \geq 1$,

$$\psi_n(x) = \begin{cases} 1, & \text{for } 0 < x_d < 1/2, \\ 0, & \text{for } x_d > 1. \end{cases}$$

Then, we notice that ψ_0 satisfies (3.109). For r small enough, we have

$$\psi_n \varphi_n = \varphi_n, \text{ for all } n \geq 0. \quad (3.110)$$

For $n = 0$, let L_0 be a uniformly elliptic parabolic operator on \mathbb{R}^d with bounded, $C_\rho^\alpha(\mathbb{H}_T)$ -Hölder continuous coefficients, such that L_0 agrees with L on the support of ψ_0 . Define the operator

$$M_0 : C_\rho^\alpha([0, T] \times \mathbb{R}^d) \rightarrow \hat{C}_\rho^{2+\alpha}([0, T] \times \mathbb{R}^d),$$

be the inverse of L_0 , as given by [24, Theorem 8.9.2]. For $n = 1, 2, \dots$, let L_n be the degenerate-parabolic operator obtained by freezing the variable coefficients $a_{ij}(t, x)$, $b_i(t, x)$ and $c(t, x)$ at $(0, x^n)$. Define the operator

$$M_n : C_s^\alpha(\bar{\mathbb{H}}_T) \rightarrow \hat{C}_s^{2+\alpha}(\bar{\mathbb{H}}_T),$$

be the inverse of L_n , as given by Proposition A.1. Define the operator

$$M : \mathcal{C}^\alpha(\bar{\mathbb{H}}_T) \rightarrow \hat{\mathcal{C}}^{2+\alpha}(\bar{\mathbb{H}}_T)$$

by setting

$$Mf := \sum_{n=0}^{\infty} \varphi_n M_n \psi_n f, \quad \text{for } f \in \mathcal{C}^\alpha(\bar{\mathbb{H}}_T).$$

Our goal is to show that (3.108) holds, for small enough r and T . We have

$$\begin{aligned} LMf - f &= \sum_{n=0}^{\infty} L\varphi_n M_n \psi_n f - f \\ &= \sum_{n=0}^{\infty} \varphi_n LM_n \psi_n f + \sum_{n=0}^{\infty} [L, \varphi_n] M_n \psi_n f - f, \end{aligned}$$

where $[L, \varphi_n]$ is given by (3.64). Denoting

$$u_n := M_n \psi_n f, \quad \text{for } n = 0, 1, 2, \dots, \quad (3.111)$$

we have

$$\begin{aligned} LM_n \psi_n f &= (L - L_n)u_n + L_n M_n \psi_n f \\ &= (L - L_n)u_n + \psi_n f, \end{aligned}$$

since $L_n M_n = I$, for all $n \geq 0$. This implies, by the identities (3.110) and $\sum_{n=0}^{\infty} \varphi_n \psi_n f = f$, that

$$LMf - f = \sum_{n=0}^{\infty} \varphi_n (L - L_n) u_n + \sum_{n=0}^{\infty} [L, \varphi_n] u_n. \quad (3.112)$$

First, we estimate the terms in the preceding equality indexed by $n = 0$. Because $L_0 = L$ on the support of ψ_0 , obviously we have $\psi_0 (L - L_0) u_0 = 0$. Next, using the identity (3.64), there is a positive constant C , depending only on K_2 in (3.97), such that

$$\begin{aligned} \|[L, \varphi_0] u_0\|_{C_\rho^\alpha([0, T] \times \mathbb{R}^d)} &\leq C \|u_0\|_{C_\rho^{1+\alpha}([0, T] \times \mathbb{R}^d)} \|\psi_0\|_{C_\rho^{2+\alpha}([0, T] \times \mathbb{R}^d)} \\ &\leq C r^{-3} \|u_0\|_{C_\rho^{1+\alpha}([0, T] \times \mathbb{R}^d)} \quad (\text{by (3.109)}). \end{aligned}$$

From the interpolation inequalities for standard Hölder spaces [24, Theorem 8.8.1], there is a positive constant m such that, for all $\varepsilon > 0$, we have

$$\|[L, \varphi_0] u_0\|_{C_\rho^\alpha([0, T] \times \mathbb{R}^d)} \leq C r^{-3} \left(\varepsilon \|u_0\|_{C_\rho^{1+\alpha}([0, T] \times \mathbb{R}^d)} + \varepsilon^{-m} \|u_0\|_{C([0, T] \times \mathbb{R}^d)} \right). \quad (3.113)$$

By [24, Theorem 8.9.2], the identity (3.110), and the definition (3.111) of u_0 , we have

$$\begin{aligned} \|u_0\|_{C_\rho^{1+\alpha}([0, T] \times \mathbb{R}^d)} &\leq C_1(r) \|\psi_0 f\|_{C_\rho^\alpha([0, T] \times \mathbb{R}^d)} \\ &\leq C_1(r) \|f\|_{\mathcal{C}^\alpha(\mathbb{H}_T)}, \end{aligned}$$

for some positive constant $C_1(r)$. From [24, Corollary 8.1.5], there is a constant C , depending only on K_2 , T and d , such that

$$\|u_0\|_{C([0, T] \times \mathbb{R}^d)} \leq CT \|f\|_{C([0, T] \times \mathbb{R}^d)}.$$

Therefore, we obtain in (3.113), for possibly a different constant $C_1(r)$,

$$\|[L, \varphi_0] u_0\|_{C_\rho^\alpha([0, T] \times \mathbb{R}^d)} \leq C_1(r) \left(\varepsilon \|f\|_{\mathcal{C}^\alpha(\mathbb{H}_T)} + \varepsilon^{-m} T \|f\|_{C(\mathbb{H}_T)} \right). \quad (3.114)$$

Next, we estimate the terms in (3.112) indexed by $n \geq 1$. We closely follow the argument used to prove Theorem 3.8. First, we have

$$\begin{aligned} \|\varphi_n (L - L_n) u_n\|_{C_s^\alpha(\mathbb{H}_T)} &\leq [\varphi_n]_{C_s^\alpha(\mathbb{H}_T)} \|(L - L_n) u_n\|_{C([0, T] \times \text{supp } \varphi_n)} \\ &\quad + \|(L - L_n) u_n\|_{C_s^\alpha([0, T] \times \text{supp } \varphi_n)}. \end{aligned} \quad (3.115)$$

Using (3.109) and Lemma 3.2, there are positive constants m and $C_1(r)$ such that

$$[\varphi_n]_{C_s^\alpha(\mathbb{H}_T)} \|(L - L_n) u_n\|_{C([0, T] \times \text{supp } \varphi_n)} \leq C_1(r) \left(\varepsilon \|u_n\|_{C_s^{2+\alpha}(\mathbb{H}_T)} + \varepsilon^{-m} \|u_n\|_{C(\mathbb{H}_T)} \right).$$

By Proposition A.1, (3.53) and the preceding inequality, we obtain

$$[\varphi_n]_{C_s^\alpha(\mathbb{H}_T)} \|(L - L_n) u_n\|_{C([0, T] \times \text{supp } \varphi_n)} \leq C_1(r) \left(\varepsilon \|\psi_n f\|_{C_s^\alpha(\mathbb{H}_T)} + \varepsilon^{-m} T \|\psi_n f\|_{C(\mathbb{H}_T)} \right),$$

and thus,

$$[\varphi_n]_{C_s^\alpha(\mathbb{H}_T)} \|(L - L_n) u_n\|_{C([0, T] \times \text{supp } \varphi_n)} \leq C_1(r) \left(\varepsilon \|f\|_{\mathcal{C}^\alpha(\mathbb{H}_T)} + \varepsilon^{-m} T \|f\|_{C(\mathbb{H}_T)} \right). \quad (3.116)$$

By applying the same argument used to prove Claim 3.9, we find that there are positive constants C , independent of r , and $C_1(r)$, such that

$$\|(L - L_n) u_n\|_{C_s^\alpha([0, T] \times \text{supp } \varphi_n)} \leq C r^{\alpha/2} \|u_n\|_{C_s^\alpha(\mathbb{H}_T)} + C_1(r) \|u_n\|_{C(\mathbb{H}_T)}.$$

By Proposition A.1, (3.53) and the definition (3.111) of u_n , it follows that

$$\|(L - L_n) u_n\|_{C_s^\alpha([0, T] \times \text{supp } \varphi_n)} \leq C r^{\alpha/2} \|f\|_{\mathcal{C}^\alpha(\mathbb{H}_T)} + C_1(r) T \|f\|_{C(\mathbb{H}_T)}. \quad (3.117)$$

With the aid of inequalities (3.116) and (3.117), the estimate (3.115) becomes

$$\|\varphi_n(L - L_n)u_n\|_{C_s^\alpha(\overline{\mathbb{H}}_T)} \leq Cr^{\alpha/2}\|f\|_{\mathcal{C}^\alpha(\overline{\mathbb{H}}_T)} + C_1(r) \left(\varepsilon\|f\|_{\mathcal{C}^\alpha(\overline{\mathbb{H}}_T)} + \varepsilon^{-m}T\|f\|_{C(\overline{\mathbb{H}}_T)} \right). \quad (3.118)$$

Next, we estimate $[L, \varphi_n]u_n$, for $n \geq 1$, by employing a method similar to that used to estimate the term $[L, \varphi_0]u_0$. Using the identity (3.64) there is a positive constant C , depending only on K appearing in (2.13) and (2.14), such that

$$\|[L, \varphi_n]u_n\|_{C_s^\alpha([0, T] \times \mathbb{H})} \leq Cr^{-3}\|u_n\|_{C_s^{1+\alpha}([0, T] \times \mathbb{H})} \quad (\text{by (3.109)}).$$

From Lemma 3.2, there is a positive constant m such that, for all $\varepsilon \in (0, 1)$, we have

$$\|[L, \varphi_n]u_n\|_{C_s^\alpha([0, T] \times \mathbb{H})} \leq Cr^{-3} \left(\varepsilon\|u_n\|_{C_s^{1+\alpha}([0, T] \times \mathbb{H})} + \varepsilon^{-m}\|u_n\|_{C([0, T] \times \mathbb{H})} \right).$$

According to Proposition A.1 and (3.53), there is a constant $C_1(r)$ so that

$$\|[L, \varphi_n]u_n\|_{C_s^\alpha([0, T] \times \mathbb{H})} \leq C_1(r) \left(\varepsilon\|f\|_{\mathcal{C}^\alpha(\overline{\mathbb{H}}_T)} + \varepsilon^{-m}T\|f\|_{C(\overline{\mathbb{H}}_T)} \right). \quad (3.119)$$

Combining inequalities (3.114), (3.118) and (3.119), and using the fact that at most N balls in the covering have non-empty intersection, the identity (3.112) yields

$$\|LMf - f\|_{\mathcal{C}^{2+\alpha}(\overline{\mathbb{H}}_T)} \leq Cr^{\alpha/2}\|f\|_{\mathcal{C}^\alpha(\overline{\mathbb{H}}_T)} + C_1(r) \left(\varepsilon\|f\|_{\mathcal{C}^\alpha(\overline{\mathbb{H}}_T)} + \varepsilon^{-m}T\|f\|_{C(\overline{\mathbb{H}}_T)} \right),$$

where C is a positive constant independent of r , while $C_1(r)$ may depend on r . By choosing small enough r , then small enough ε , and then small enough T , in that order, we find a positive constant $C_0 < 1$ such that

$$\|LMf - f\|_{\mathcal{C}^\alpha(\overline{\mathbb{H}}_T)} \leq C_0\|f\|_{\mathcal{C}^\alpha(\overline{\mathbb{H}}_T)}, \quad \forall f \in \mathcal{C}^\alpha(\overline{\mathbb{H}}_T),$$

and this gives (3.108). \square

Proof of Theorem 1.1. Uniqueness of solutions follows from Proposition 3.7.

We notice that $\mathcal{C}_p^\alpha(\overline{\mathbb{H}}_T) \subset \mathcal{C}^\alpha(\overline{\mathbb{H}}_T)$ and $\mathcal{C}_p^{2+\alpha}(\overline{\mathbb{H}}) \subset \mathcal{C}^{2+\alpha}(\overline{\mathbb{H}})$. Let \tilde{L} be any operator satisfying Hypothesis 3.16. Let $\{\varphi_n\}_{n \geq 1}$ be a sequence of non-negative, smooth cut-off functions such that

$$0 \leq \varphi_n \leq 1, \quad \varphi_n|_{B_n} = 1, \quad \text{and } \varphi_n|_{B_{2n}^c} = 0.$$

We define

$$L_n := \varphi_n L + (1 - \varphi_n)\tilde{L}, \quad \forall n \geq 1.$$

Then, each L_n satisfies Hypothesis 3.16 and, by Proposition 3.19, there exists a unique solution $u_n \in \mathcal{C}^{2+\alpha}(\overline{\mathbb{H}}_T)$ to (1.1) with $L = L_n$. By Lemma 3.18, each solution u_n satisfies the global estimate

$$\|u_n\|_{\mathcal{C}^{2+\alpha}(\overline{\mathbb{H}}_T)} \leq C \left(\|f\|_{\mathcal{C}_p^\alpha(\overline{\mathbb{H}}_T)} + \|g\|_{\mathcal{C}_p^{2+\alpha}(\overline{\mathbb{H}})} \right). \quad (3.120)$$

For any bounded subdomain $U \subset \mathbb{H}$ and denoting $U_T = (0, T) \times U$, the parabolic analogue, $C_\rho^{2+\alpha}(\bar{U}_T) \hookrightarrow C_\rho^2(\bar{U}_T) \equiv C^{1,2}(\bar{U}_T)$, of the compact embedding [1, Theorem 1.31 (4)] of standard Hölder spaces, $C^{2+\alpha}(\bar{U}) \hookrightarrow C^2(\bar{U})$, implies that the sequence $\{u_n\}_{n \geq 1}$ converges strongly in $C^{1,2}(\bar{U}_T)$ to the limit $u \in C^{1,2}(U_T)$, that is, $u_n \rightarrow u$ in $C^{1,2}(U_T)$, as $n \rightarrow \infty$ for every bounded subdomain $U \subset \mathbb{H}$. It is now easily seen that u solves (1.1). By the Arzelà-Ascoli Theorem, we obtain that $u \in \mathcal{C}^{2+\alpha}(\overline{\mathbb{H}}_T)$ and satisfies (1.4). \square

APPENDIX A. EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR A DEGENERATE-PARABOLIC OPERATOR WITH CONSTANT COEFFICIENTS

In order to derive the local a priori boundary estimates in Theorem 3.8, we need an analogue of [3, Theorem I.1.1] when the coefficients of our operator L , a_{ij} , b_i and c , are assumed *constant*. To emphasize this fact in this appendix, we denote our parabolic operator by

$$-L_0 u := -u_t + \sum_{i,j=1}^d x_d a_{ij} u_{x_i x_j} + \sum_{i=1}^d b_i u_{x_i} + cu \quad \text{on } (0, T) \times \mathbb{H}. \quad (\text{A.1})$$

We now have the following analogue of [3, Theorem I.1.1].

Proposition A.1 (Existence and uniqueness of solutions for a degenerate parabolic operator with constant coefficients). *Let K , δ and ν be positive constants such that*

$$\sum_{i,j=1}^d a_{ij} \eta_i \eta_j \geq \delta \|\eta\|^2, \quad \forall \eta \in \mathbb{R}^d, \quad (\text{A.2})$$

$$b_d \geq \nu, \quad (\text{A.3})$$

$$|a_{ij}|, |b_i|, |c| \leq K, \quad \forall 1 \leq i, j \leq d. \quad (\text{A.4})$$

Let k be a nonnegative integer, $T > 0$, and $\alpha \in (0, 1)$. Assume that $f \in C_s^{k, \alpha}(\overline{\mathbb{H}}_T)$ and $g \in C_s^{k, 2+\alpha}(\overline{\mathbb{H}})$ with both f and g compactly supported in $\overline{\mathbb{H}}_T$ and $\overline{\mathbb{H}}$, respectively. Then, the inhomogeneous initial value problem

$$\begin{cases} L_0 u = f & \text{on } (0, T) \times \mathbb{H}, \\ u(0, \cdot) = g & \text{on } \overline{\mathbb{H}}. \end{cases} \quad (\text{A.5})$$

admits a unique solution $u \in C_s^{k, 2+\alpha}(\overline{\mathbb{H}}_T)$. Moreover, there exists a positive constant $C = C(T, K, \delta, \nu, \alpha, d, k)$ such that

$$\|u\|_{C_s^{k, 2+\alpha}(\overline{\mathbb{H}}_T)} \leq C \left(\|f\|_{C_s^{k, \alpha}(\overline{\mathbb{H}}_T)} + \|g\|_{C_s^{k, 2+\alpha}(\overline{\mathbb{H}})} \right). \quad (\text{A.6})$$

Proof. The proof follows by adapting the proof of [3, Theorem I.1.1]. Because the proof of [3, Theorem I.1.1] is lengthy, we only outline the modifications, noting that these modifications are straightforward. We remark that there is no simple change of variables that can be applied in order to bring the constant coefficients equation (A.5) to the form of the model operator defined in [3, p. 901]. Another difficulty is that our interpolation inequalities (Lemma 3.2) do not allow us to treat the first order derivatives, u_{x_i} , in (1.2) as lower order terms: in order to do that, we would need to have

$$\|u_{x_i}\|_{C_s^\alpha(\overline{\mathbb{H}}_T)} \leq \varepsilon \|u\|_{C_s^{2+\alpha}(\overline{\mathbb{H}}_T)} + C\varepsilon^{-m} \|u\|_{C(\overline{\mathbb{H}}_T)},$$

instead of the interpolation inequality (3.8). On the other hand, by simple changes of variables which we describe below and which preserve the domain \mathbb{H} and its boundary $\partial\mathbb{H}$, problem (A.5) can be simplified to

$$-L_0 u = -u_t + x_d \sum_{i=1}^d u_{x_i x_i} + \sum_{i=1}^d b_i u_{x_i} \quad \text{on } (0, T) \times \mathbb{H}, \quad (\text{A.7})$$

where the coefficient $b_d > 0$ remains unchanged. In addition, the possibly new constant coefficients b_i are bounded in absolute value by constants which depend only on δ in (A.2) and K in (A.4).

The simple changes of variables are as follows. As usual, we eliminate the zeroth order term cu by multiplying u by e^{ct} and so we may assume without loss of generality that $c = 0$ in (A.1). We define a function \tilde{u} on $(0, T) \times \mathbb{H}$ by choosing $y = (y_1, \dots, y_d)$ and

$$(y_1, \dots, y_d) := (x_1 + \alpha_1 x_d, \dots, x_{d-1} + \alpha_{d-1} x_d, x_d), \quad \text{where } \alpha_i := -\frac{a_{id}}{a_{dd}}, \quad 1 \leq i \leq d-1,$$

and

$$\tilde{u}(t, y) := u(t, x).$$

Note that $a_{dd} > \delta$, by choosing $\eta = (0, 0, \dots, 1)$ in (A.2). By direct calculations, we obtain (omitting the arguments of the functions u and \tilde{u} for brevity),

$$\begin{aligned} u_{x_i} &= \tilde{u}_{y_i}, \quad \forall i \neq d, \\ u_{x_d} &= \sum_{k \neq d} \alpha_k \tilde{u}_{y_k} + \tilde{u}_{y_d}, \\ u_{x_i x_j} &= \tilde{u}_{y_i y_j}, \quad \forall i, j \neq d, \\ u_{x_i x_d} &= \sum_{k \neq d} \alpha_k \tilde{u}_{y_i y_k} + \tilde{u}_{y_i y_d}, \quad \forall i \neq d, \\ u_{x_d x_d} &= \sum_{k, l \neq d} \alpha_k \alpha_l \tilde{u}_{y_k y_l} + 2 \sum_{k \neq d} \alpha_k \tilde{u}_{y_k y_d} + \tilde{u}_{y_d y_d}, \end{aligned}$$

from where it follows that,

$$\begin{aligned} Lu &= \tilde{u}_t - y_d a_{dd} \tilde{u}_{y_d y_d} - 2y_d \sum_{i \neq d} (a_{dd} \alpha_i + a_{id}) \tilde{u}_{y_i y_d} - y_d \sum_{i, j \neq d} (a_{dd} \alpha_i \alpha_j + a_{ij} + a_{id} \alpha_j) \tilde{u}_{y_i y_j} \\ &\quad - \sum_{i \neq d} (b_i + \alpha_i b_d) \tilde{u}_{y_i} - b_d \tilde{u}_{y_d}. \end{aligned}$$

Since $\alpha_i = a_{id}/a_{dd}$ for all $i \neq d$, we obtain

$$\begin{aligned} a_{dd} \alpha_i + a_{id} &= 0, \quad \forall i \neq d, \\ a_{dd} \alpha_i \alpha_j + a_{ij} + a_{id} \alpha_j &= a_{ij}, \quad \forall i, j \neq d, \end{aligned}$$

and so problem (A.5) is reduced to the study of the operator \tilde{L}_0 on $(0, T) \times \mathbb{H}$ given by

$$\tilde{L}_0 \tilde{u} := \tilde{u}_t - y_d a_{dd} \tilde{u}_{y_d y_d} - y_d \sum_{i, j \neq d} a_{ij} \tilde{u}_{y_i y_j} - \sum_{i \neq d} (b_i + \alpha_i b_d) \tilde{u}_{y_i} - b_d \tilde{u}_{y_d}.$$

Because the $(d-1) \times (d-1)$ matrix $\bar{a} := (a_{ij})_{i, j=1, \dots, d-1}$ is positive-definite and symmetric, there is an orthogonal matrix P such that $P^* \bar{a} P = D$, where $D := \text{diag}(\lambda_1, \dots, \lambda_{d-1})$ and λ_i , $i = 1, \dots, d-1$, are the (positive) eigenvalues of \bar{a} . By setting

$$Q := \begin{pmatrix} PD^{-1/2} & 0 \\ 0 & a_{dd}^{-1/2} \end{pmatrix}, \quad \text{where } D^{-1/2} = \text{diag}(\lambda_1^{-1/2}, \dots, \lambda_{d-1}^{-1/2}),$$

we notice that

$$Q^* \begin{pmatrix} \bar{a} & 0 \\ 0 & a_{dd} \end{pmatrix} Q = I_d,$$

where I_d is the $d \times d$ identity matrix. Proceeding as in the [19, Proof of Lemma 6.1], we choose $z := yQ$ and

$$\bar{u}(t, z) := \tilde{u}(t, y), \quad \forall (t, y) \in (0, T) \times \mathbb{H}.$$

Then, direct calculations show that problem (A.5) is reduced to the study of the operator \bar{L}_0 on $(0, T) \times \mathbb{H}$ given by

$$-\bar{L}_0 \bar{u} := -\bar{u}_t + z_d \sum_{i=1}^d \bar{u}_{z_i z_i} + \sum_{i=1}^d \bar{b}_i \bar{u}_{z_i},$$

where the constant coefficients \bar{b}_i may differ from the coefficients b_i , for $i \neq d$, and the coefficient $\bar{b}_d := \sqrt{a_{dd}} b_d > 0$. Therefore, for the remainder of this section, we may assume without loss of generality that L_0 is of the simpler form (A.7).

The primary change required in the proof of [3, Theorem I.1.1] lies in [3, §I.4]. The arguments in the remainder of [3, Part I] adapt almost line by line to our model operator (A.7). The goal in [3, §I.4] is to derive local derivative estimates and this is achieved by applying a comparison principle with barrier functions. First, we need to adapt the definition of the barrier function [3, Definition I.4.1] to one which is suitable for use with (A.7).

Definition A.2. Let $0 < t_1 < t_2$. We say φ is a *barrier function* for L_0 when $t \in [t_1, t_2]$, if there are positive constants C and c such

$$L_0 \varphi > -C x_d \varphi^2 + c \varphi^{3/2} + c. \quad (\text{A.8})$$

The barrier functions in [3, Theorem I.4.5 & Theorem I.4.8] are barrier functions in the sense of Definition A.2, also. The barrier function constructed in [3, Theorem I.4.6] needs modification because the coefficients b_i , $i = 1, \dots, d-1$, are non-zero in general, unlike in [3, Part I]. We have the following modification.

Claim A.3. Assume $i \neq d$. For any $\gamma < 1$ as in [3, Definition I.4.2], there are a positive constant b depending only on $|b_i|$, and a positive constant Δ , depending only on $|b_i|$, b and γ such that, for any $t_0 \geq 0$,

$$\varphi_i(t, x) := \frac{1}{(1 + x_i - b(t - t_0))^2} + \frac{1}{(1 - x_i - b(t - t_0))^2} \quad (\text{A.9})$$

is a valid barrier function satisfying (A.8), for all $t \in [t_0, t_0 + \Delta t]$.

Proof of Claim A.3. It suffices to consider separately the terms $^+ \varphi_i$ and $^- \varphi_i$ defined by

$$^\pm \varphi := \frac{1}{(1 \pm x_i - b(t - t_0))^2},$$

because the barrier functions form a cone by [3, Theorem I.4.4]. We prove that $^+ \varphi_i$ satisfies (A.8), and the proof follows similarly for $^- \varphi_i$. We denote for simplicity $\varphi := ^+ \varphi_i$. By direct calculation, we obtain

$$\begin{aligned} \varphi_t &= 2b\varphi^{3/2}, \\ \varphi_{x_i} &= -2\varphi^{3/2}, \\ \varphi_{x_i x_i} &= 6\varphi^2, \end{aligned}$$

while $\varphi_{x_j} = 0$ and $\varphi_{x_j x_k} = 0$, unless $j = k = i$. Then, we have

$$L_0 \varphi = 2(b + b_i)\varphi^{3/2} - 6x_d \varphi^2.$$

We impose $1 - b(t - t_0) \geq \gamma$, for all $t \in [t_0, t_0 + \Delta]$, so we choose $\Delta < (1 - \gamma)/b$. By choosing $b = |b_i| + 1$, we can find $C > 0$ and $c > 0$ such that

$$L_0 \varphi \geq -x_d C \varphi^2 + c \varphi^{3/2} + c,$$

and so φ satisfies the requirement (A.8), for all $t \in [t_0, t_0 + \Delta]$. \square

Next, the arguments in [3, §I.5] adapt to our framework with the following observation. Because our barrier functions (A.9) are not defined for all $t \in [0, 1]$, we cover first the interval $[0, 1]$ by a finite number of intervals of length Δ , as given in Claim A.3, and we apply the maximum principle on each of the resulting subintervals. This will yield local estimates analogous to [3, Theorem I.5.1, I.5.4 & Corollary I.5.7], on the small time subintervals of the finite covering. By combining the local derivative estimates over each subinterval, we obtain the required local estimates for all $t \in [0, 1]$. \square

APPENDIX B. PROOFS OF LEMMA 3.11 AND PROPOSITION 3.12

We begin with the

The proof of Lemma 3.11. The proof follows the argument used to prove [24, Lemmas 9.2.1 & 8.9.1], only we are careful about the dependencies of the constants appearing in the estimates on δ_1 and K_1 , given by (3.76) and (3.77), respectively. Let U be an orthogonal matrix such that $A = U \text{diag}(\lambda_i) U^T$, where $\lambda_i \in [\delta_1, K_1]$ are the eigenvalues of the symmetric, positive definite matrix, (a_{ij}) . We denote $B = U \text{diag}(\sqrt{\lambda_i}) U^*$ and $v(t, x) = u(t, Bx)$, $\bar{f}(t, x) = f(t, Bx)$, and $\bar{g}(x) = g(Bx)$. Then, by defining $w(t, x) := e^{-t} v(t, x)$, we see that $w \in C_\rho^{2+\alpha}([0, T] \times \mathbb{R}^d)$ solves the inhomogeneous heat equation,

$$\begin{cases} w_t - \Delta w + w = e^t \bar{f} & \text{on } (0, T] \times \mathbb{R}^d, \\ w(0, \cdot) = \bar{g} & \text{on } \mathbb{R}^d. \end{cases}$$

By applying [24, Theorem 9.2.1] to w , we obtain a constant $\bar{N}_1 = \bar{N}_1(\alpha, d)$ such that

$$\|w\|_{C_\rho^{2+\alpha}([0, T] \times \mathbb{R}^d)} \leq \bar{N}_1 \left(\|e^t \bar{f}\|_{C_\rho^\alpha([0, T] \times \mathbb{R}^d)} + \|\bar{g}\|_{C_\rho^{2+\alpha}(\mathbb{R}^d)} \right),$$

which gives us for $v(t, x) = e^t w(t, x)$ the estimate

$$\|v\|_{C_\rho^{2+\alpha}([0, T] \times \mathbb{R}^d)} \leq N_1 \left(\|\bar{f}\|_{C_\rho^\alpha([0, T] \times \mathbb{R}^d)} + \|\bar{g}\|_{C_\rho^{2+\alpha}(\mathbb{R}^d)} \right), \quad (\text{B.1})$$

where now $N_1 = N_1(\alpha, d, T)$.

To obtain (3.82) from (B.1), we need the following

Claim B.1. *There is a positive constant $C = C(d)$, such that for any $w_1 \in C_\rho^\alpha([0, T] \times \mathbb{R}^d)$ and any symmetric, positive-definite $d \times d$ -matrix, M , with eigenvalues in $[\lambda_{\min}, \lambda_{\max}]$, where $\lambda_{\max} > \lambda_{\min} > 0$, we have*

$$\|w_1\|_{C_\rho^\alpha([0, T] \times \mathbb{R}^d)} \leq C(1 + \lambda_{\min}^{-\alpha}) \|w_2\|_{C_\rho^\alpha([0, T] \times \mathbb{R}^d)}, \quad (\text{B.2})$$

$$\|w_2\|_{C_\rho^\alpha([0, T] \times \mathbb{R}^d)} \leq C(1 + \lambda_{\max}^\alpha) \|w_1\|_{C_\rho^\alpha([0, T] \times \mathbb{R}^d)}, \quad (\text{B.3})$$

where $w_2(t, x) := w_1(t, Mx)$.

Proof of Claim B.1. We first prove (B.2). Obviously, we have

$$\|w_1\|_{C([0, T] \times \mathbb{R}^d)} = \|w_2\|_{C([0, T] \times \mathbb{R}^d)}. \quad (\text{B.4})$$

Next, it suffices to consider $|w_1(P^1) - w_1(P^2)|/\rho^\alpha(P, Q)$, for points $P_i = (t^i, x^i) \in [0, T] \times \mathbb{R}^d$, $i = 1, 2$, where only one of the coordinates differs. Notice that when $x^1 = x^2$, then

$$\frac{|w_1(P^1) - w_1(P^2)|}{\rho^\alpha(P^1, P^2)} = \frac{|w_2(P^1) - w_2(P^2)|}{\rho^\alpha(P^1, P^2)},$$

because the transformation $w_2(t, x) := w_1(t, Mx)$ acts only on the spatial variables. Therefore, we have

$$\frac{|w_1(P^1) - w_1(P^2)|}{\rho^\alpha(P^1, P^2)} \leq [w_2]_{C_\rho^\alpha([0, T] \times \mathbb{R}^d)}, \quad (\text{B.5})$$

Next, we consider the case $t^1 = t^2 = t$. Then, we have by writing $w_1(t, x) = w_2(t, M^{-1}x)$,

$$\frac{|w_1(P^1) - w_1(P^2)|}{\rho^\alpha(P^1, P^2)} = \frac{|w_2(t, M^{-1}x^1) - w_2(t, M^{-1}x^2)|}{|M(M^{-1}x^1 - M^{-1}x^2)|^\alpha}$$

Using the fact that M is a symmetric, positive-definite matrix with eigenvalues in the range $[\lambda_{\min}, \lambda_{\max}]$, it follows

$$|M(M^{-1}x^1 - M^{-1}x^2)| \geq \lambda_{\min}|M^{-1}x^1 - M^{-1}x^2|, \quad \forall x^1, x^2 \in \mathbb{R}^d,$$

and so, by the preceding two inequalities, we have

$$\frac{|w_1(P^1) - w_1(P^2)|}{\rho^\alpha(P^1, P^2)} \leq \lambda_{\min}^{-\alpha} [w_2]_{C_\rho^\alpha([0, T] \times \mathbb{R}^d)}. \quad (\text{B.6})$$

Combining inequalities (B.4), (B.5) and (B.6), we obtain (B.2).

To obtain (B.3), we apply (B.2) to w_2 in place of w_1 . Then, the matrix M is replaced by the symmetric, positive-definite matrix M^{-1} with eigenvalues in $[\lambda_{\max}^{-1}, \lambda_{\min}^{-1}]$. Therefore, λ_{\min}^{-1} in (B.2) is replaced by λ_{\max} , and thus, we obtain (B.3). \square

Notice that B is a symmetric, positive-definite matrix with eigenvalues in $[\sqrt{\delta_1}, \sqrt{K_1}]$. Since $v(t, x) = u(t, Bx)$, we may apply (B.2) with $w_1 = u$ and $w_2 = v$ and $M = B$ to obtain

$$\|u\|_{C_\rho^\alpha([0, T] \times \mathbb{R}^d)} \leq C(1 + \delta_1^{-\alpha/2}) \|v\|_{C_\rho^\alpha([0, T] \times \mathbb{R}^d)}. \quad (\text{B.7})$$

Because $v_t(t, x) = u_t(t, Bx)$, we have as above

$$\|u_t\|_{C_\rho^\alpha([0, T] \times \mathbb{R}^d)} \leq C(1 + \delta_1^{-\alpha/2}) \|v_t\|_{C_\rho^\alpha([0, T] \times \mathbb{R}^d)}. \quad (\text{B.8})$$

To evaluate u_{x_i} , we denote by L^i the i -th row of the matrix B^{-1} . Then, we have

$$u_{x_i} = L^i \nabla v,$$

and so,

$$\|u_{x_i}\|_{C([0, T] \times \mathbb{R}^d)} \leq \delta_1^{-1/2} \|\nabla v\|_{C([0, T] \times \mathbb{R}^d)}.$$

where we have use the fact that B^{-1} is a symmetric, positive-definite matrix and the eigenvalues of B^{-1} are in $[K_1^{-1/2}, \delta_1^{-1/2}]$. Applying inequality B.2 to u_{x_i} , we obtain as above

$$\|u_{x_i}\|_{C_\rho^\alpha([0, T] \times \mathbb{R}^d)} \leq C\delta_1^{-1/2}(1 + \delta_1^{-\alpha/2}) \|v_{x_i}\|_{C_\rho^\alpha([0, T] \times \mathbb{R}^d)}, \quad (\text{B.9})$$

and similarly, it follows for $u_{x_i x_j}$

$$\|u_{x_i x_j}\|_{C_\rho^\alpha([0, T] \times \mathbb{R}^d)} \leq \delta_1^{-1}(1 + \delta_1^{-\alpha/2}) \|v_{x_i x_j}\|_{C_\rho^\alpha([0, T] \times \mathbb{R}^d)}. \quad (\text{B.10})$$

Applying (B.3) for $\bar{f}(t, x) = f(t, Bx)$ with $w_1 = f$ and $w_2 = \bar{f}$ and $M = B$, we have

$$\|\bar{f}\|_{C_\rho^\alpha([0, T] \times \mathbb{R}^d)} \leq (1 + K_1^{\alpha/2}) \|f\|_{C_\rho^\alpha([0, T] \times \mathbb{R}^d)}, \quad (\text{B.11})$$

Similarly, for $\bar{g}(x) = g(Bx)$, we obtain

$$\begin{aligned} \|\bar{g}\|_{C_\rho^\alpha(\mathbb{R}^d)} &\leq (1 + K_1^{\alpha/2}) \|g\|_{C_\rho^\alpha([0,T] \times \mathbb{R}^d)}, \\ \|\bar{g}_{x_i}\|_{C_\rho^\alpha(\mathbb{R}^d)} &\leq K_1^{1/2} (1 + K_1^{\alpha/2}) \|g_{x_i}\|_{C_\rho^\alpha(\mathbb{R}^d)}, \\ \|\bar{g}_{x_i x_j}\|_{C_\rho^\alpha(\mathbb{R}^d)} &\leq K_1 (1 + K_1^{\alpha/2}) \|g_{x_i x_j}\|_{C_\rho^\alpha(\mathbb{R}^d)}. \end{aligned} \quad (\text{B.12})$$

By combining the inequalities (B.7), (B.8), (B.9), (B.10), (B.11) and (B.12) in (B.1), we obtain (3.82). \square

Next, we give the proof of Proposition 3.12. The estimate (3.86) is obtained exactly as in the proof of [24, Theorems 9.2.2 & 8.9.2] using Lemma 3.11, except that we again provide the details in order to obtain the precise dependencies of the coefficients.

Proof of Proposition 3.12. Due to the classical interpolation inequalities [24, Theorem 8.8.1] and the classical maximum principle for unbounded domains [24, Corollary 8.1.5], it suffices to prove that the estimate (3.86) holds with

$$[u_t]_{C_\rho^\alpha([0,T] \times \mathbb{R}^d)} \quad \text{and} \quad [u_{x_i x_j}]_{C_\rho^\alpha([0,T] \times \mathbb{R}^d)}$$

on the left hand side of the inequality. We will prove this for the $C_\rho^\alpha([0,T] \times \mathbb{R}^d)$ -seminorm of u_t , but the same argument can be applied for the $C_\rho^\alpha([0,T] \times \mathbb{R}^d)$ -seminorm of $u_{x_i x_j}$.

For simplicity of notation, we denote $Q := (0, T) \times \mathbb{R}^d$, and we omit the subscript ρ in the definition of the Hölder spaces. We also use the simplified notation

$$[u]_{C^{2+\alpha}(\bar{Q})} := [u_t]_{C^\alpha(\bar{Q})} + [u_{x_i x_j}]_{C^\alpha(\bar{Q})}. \quad (\text{B.13})$$

Let $u \in C^{2+\alpha}(\bar{Q})$ be a solution to Problem 3.81. Then,

$$\bar{u} := e^{-\lambda_1 t} u \quad (\text{B.14})$$

is in $C^{2+\alpha}(\bar{Q})$ and it solves

$$\begin{cases} (-\bar{L} - \lambda_1) \bar{u} = -e^{-\lambda_1 t} f & \text{on } (0, T) \times \mathbb{R}^d, \\ \bar{u}(0, \cdot) = g & \text{on } \mathbb{R}^d, \end{cases}$$

where λ_1 is the upper bound on the zeroth order coefficient, \bar{c} , assumed in (3.78). We may apply [24, Corollary 8.1.5], because the zeroth order term of the parabolic operator $-\bar{L} - \lambda_1$ is nonpositive, and we obtain

$$\|\bar{u}\|_{C(\bar{Q})} \leq T \|e^{-\lambda_1 t} f\|_{C(\bar{Q})} + \|g\|_{C(\bar{Q})} \leq T \|f\|_{C(\bar{Q})} + \|g\|_{C(\bar{Q})}.$$

Thus, it follows by (B.14)

$$\|u\|_{C(\bar{Q})} \leq e^{\lambda_1 T} \left(T \|f\|_{C(\bar{Q})} + \|g\|_{C(\bar{Q})} \right). \quad (\text{B.15})$$

Let $z_1, z_2 \in [0, T] \times \mathbb{R}^d$ be two points such that

$$\frac{|u_t(z_1) - u_t(z_2)|}{\rho^\alpha(z_1, z_2)} \geq \frac{1}{2} [u_t]_{C^\alpha(\bar{Q})}. \quad (\text{B.16})$$

Let $\gamma > 0$ be a constant which will be suitably chosen below. We consider two cases.

Case 1 ($\rho(z_1, z_2) \geq \gamma$). Then, we have

$$[u_t]_{C^\alpha(\bar{Q})} \leq 2\gamma^{-\alpha} |u_t|_{C(\bar{Q})},$$

and, by [24, Theorem 8.8.1, Inequality (8.8.1)], it follows, for all $\varepsilon > 0$,

$$[u_t]_{C^\alpha(\bar{Q})} \leq 2\gamma^{-\alpha} \left(\varepsilon [u]_{C^{2+\alpha}(\bar{Q})} + C\varepsilon^{-\alpha/2} |u|_{C(\bar{Q})} \right).$$

By choosing $\varepsilon := \gamma^\alpha/8$ and by inequality (B.15), we obtain

$$[u_t]_{C^\alpha(\bar{Q})} \leq \frac{1}{4} [u]_{C^{2+\alpha}(\bar{Q})} + C\gamma^{-(\alpha+\alpha^2/2)} e^{\lambda_1 T} \left(T\|f\|_{C(\bar{Q})} + \|g\|_{C(\bar{Q})} \right), \quad (\text{B.17})$$

where the constant $C = C(d, \alpha)$.

Case 2 ($\rho(z_1, z_2) < \gamma$). We denote $z = (t, x)$. Let $\zeta : \mathbb{R}^{d+1} \rightarrow [0, 1]$ be a smooth cutoff function such that

$$\zeta(z) = 1, \text{ if } \rho(z, z_1) \leq 1, \text{ and } \zeta(z) = 0, \text{ if } \rho(z, z_1) \geq 2,$$

and we define φ by

$$\varphi(z) := \zeta((t - t_1)/\gamma^2, (x - x_1)/\gamma) \quad \forall z \in \mathbb{R}^{d+1},$$

so that,

$$\varphi(z) = 1, \text{ if } \rho(z, z_1) \leq \gamma, \text{ and } \varphi(z) = 0, \text{ if } \rho(z, z_1) \geq 2\gamma, \quad (\text{B.18})$$

It is straightforward to see that φ satisfies

$$\|\varphi\|_{C^{2+\alpha}(\mathbb{R}^{d+1})} \leq C \left(1 + \gamma^{-(2+\alpha)} \right), \quad (\text{B.19})$$

where C is a positive constant. Since $z_2 \in \{\varphi = 1\}$, we obtain by (B.16)

$$[u_t]_{C^\alpha(\bar{Q})} \leq 2 \frac{|u_t(z_1) - u_t(z_2)|}{\rho^\alpha(z_1, z_2)} \leq 2[(u\varphi)_t]_{C^\alpha(\bar{Q})}. \quad (\text{B.20})$$

Let \bar{L}_0 denote the differential operator, with constant coefficients, of the type considered in Lemma 3.11

$$-\bar{L}_0 = -\partial_t + \sum_{i,j=1}^d \bar{a}_{ij}(z_1) \partial_{x_i x_j}. \quad (\text{B.21})$$

Estimate (3.82) shows that there are constants $p_1 = p_1(\alpha)$ and $C = C(d, \alpha, T)$ such that

$$[(u\varphi)_t]_{C^\alpha(\bar{Q})} \leq C \left(1 + \delta_1^{-p_1} + K_1^{p_1} \right) \left(\|\bar{L}_0(u\varphi)\|_{C^\alpha(\bar{Q})} + \|g\varphi\|_{C^{2+\alpha}(\{0\} \times \mathbb{R}^d)} \right). \quad (\text{B.22})$$

By (B.19), we obtain

$$\|g\varphi\|_{C^{2+\alpha}(\{0\} \times \mathbb{R}^d)} \leq C \left(1 + \gamma^{-(2+\alpha)} \right) \|g\|_{C^{2+\alpha}(\mathbb{R}^d)}. \quad (\text{B.23})$$

By writing $\bar{L}_0(u\varphi) = L(u\varphi) + (\bar{L}_0 - L)(u\varphi)$, we have

$$\bar{L}_0(u\varphi) = L(u\varphi) + \sum_{i,j=1}^d (\bar{a}_{ij}(z) - \bar{a}_{ij}(z_1)) (u\varphi)_{x_i x_j} + \sum_{i=1}^d \bar{b}_i(z) (u\varphi)_{x_i} + \bar{c}(z) u\varphi. \quad (\text{B.24})$$

We may write

$$L(u\varphi) = \varphi Lu + \sum_{i,j=1}^d \bar{a}_{ij}(z) \varphi_{x_j} u_{x_i} + \left(\sum_{i,j=1}^d \bar{a}_{ij}(z) \varphi_{x_i x_j} + \sum_{i=1}^d \bar{b}_i(z) \varphi_{x_i} + \bar{c}(z) \right) u$$

and so, by (B.19) and (3.77), we obtain there is a positive constant $C = C(d)$ such that

$$\begin{aligned} \|L(u\varphi)\|_{C^\alpha(\bar{Q})} &\leq C \left(1 + \gamma^{-(2+\alpha)}\right) \|Lu\|_{C^\alpha(\bar{Q})} \\ &\quad + CK_1 \left(1 + \gamma^{-(2+\alpha)}\right) \left(\|u_{x_i}\|_{C^\alpha(\bar{Q})} + \|u\|_{C^\alpha(\bar{Q})}\right). \end{aligned} \quad (\text{B.25})$$

Notice that we may write the difference as

$$\begin{aligned} \bar{L}_0(u\varphi) - L(u\varphi) &= \sum_{i,j=1}^d (\bar{a}_{ij}(z) - \bar{a}_{ij}(z_1)) \varphi u_{x_i x_j} \\ &\quad + \sum_{i=1}^d \left(\sum_{j=1}^d (\bar{a}_{ij}(z) - \bar{a}_{ij}(z_1)) \varphi_{x_j} + \bar{b}_i(z) \varphi \right) u_{x_i} \\ &\quad + \left(\sum_{i,j=1}^d (\bar{a}_{ij}(z) - \bar{a}_{ij}(z_1)) (\varphi)_{x_i x_j} + \sum_{i=1}^d \bar{b}_i(z) \varphi_{x_i} + \bar{c}(z) \varphi \right) u. \end{aligned}$$

By (3.77), (B.18) and (B.19), we obtain

$$\|(\bar{a}_{ij}(z) - \bar{a}_{ij}(z_1)) \varphi u_{x_i x_j}\|_{C^\alpha(\bar{Q})} \leq CK_1 \gamma^\alpha [u_{x_i x_j}]_{C^\alpha(\bar{Q})} + CK_1 (1 + \gamma^{-(2+\alpha)}) \|u_{x_i x_j}\|_{C(\bar{Q})}.$$

From an argument similar to that used to obtain (B.25), we have

$$\begin{aligned} \|\bar{L}_0(u\varphi) - L(u\varphi)\|_{C^\alpha(\bar{Q})} &\leq CK_1 \gamma^\alpha [u_{x_i x_j}]_{C^\alpha(\bar{Q})} \\ &\quad + CK_1 (1 + \gamma^{-(2+\alpha)}) \left(\|u_{x_i x_j}\|_{C(\bar{Q})} + \|u_{x_i}\|_{C^\alpha(\bar{Q})} + \|u\|_{C^\alpha(\bar{Q})} \right). \end{aligned} \quad (\text{B.26})$$

Estimates (B.25) and (B.26), give us, by (B.24),

$$\begin{aligned} \|\bar{L}_0(u\varphi)\|_{C^\alpha(\bar{Q})} &\leq C \left(1 + \gamma^{-(2+\alpha)}\right) \|Lu\|_{C^\alpha(\bar{Q})} + CK_1 \gamma^\alpha [u_{x_i x_j}]_{C^\alpha(\bar{Q})} \\ &\quad + CK_1 \left(1 + \gamma^{-(2+\alpha)}\right) \left(\|u_{x_i x_j}\|_{C(\bar{Q})} + \|u_{x_i}\|_{C^\alpha(\bar{Q})} + \|u\|_{C^\alpha(\bar{Q})} \right). \end{aligned} \quad (\text{B.27})$$

Combining the preceding inequality, estimates (B.22) and (B.23) in (B.20), and using notation (B.13), it follows

$$\begin{aligned} [u_t]_{C^\alpha(\bar{Q})} &\leq C \left(1 + \delta_1^{-p_1} + K_1^{p_1}\right) \left(\left(1 + \gamma^{-(2+\alpha)}\right) \|\bar{L}u\|_{C^\alpha(\bar{Q})} \right. \\ &\quad + K_1 \gamma^\alpha [u]_{C^{2+\alpha}(\bar{Q})} \\ &\quad + K_1 \left(1 + \gamma^{-(2+\alpha)}\right) \left(\|u_{x_i x_j}\|_{C(\bar{Q})} + \|u_{x_i}\|_{C^\alpha(\bar{Q})} + \|u\|_{C^\alpha(\bar{Q})} \right) \\ &\quad \left. + \left(1 + \gamma^{-(2+\alpha)}\right) \|g\|_{C^{2+\alpha}(\mathbb{R}^d)} \right), \end{aligned}$$

where $C = C(d, \alpha, T)$. The interpolation inequalities [24, Theorem 8.8.1] and the maximum principle [24, Corollary 8.1.5], gives us, for any $\varepsilon > 0$,

$$\begin{aligned} [u_t]_{C^\alpha(\bar{Q})} &\leq C \left(1 + \delta_1^{-p_1} + K_1^{p_1}\right) \\ &\quad \times \left[e^{\lambda_1 T} \left(1 + K_1 \varepsilon^{-m}\right) \left(1 + \gamma^{-(2+\alpha)}\right) \left(\|f\|_{C^\alpha(\bar{Q})} + \|g\|_{C^{2+\alpha}(\mathbb{R}^d)} \right) \right. \\ &\quad \left. + K_1 \left(\gamma^\alpha + \varepsilon \left(1 + \gamma^{-(2+\alpha)}\right) \right) [u]_{C^{2+\alpha}(\bar{Q})} \right], \end{aligned}$$

where $m = m(\alpha)$. We choose $\gamma \in (0, 1)$ such that

$$C \left(1 + \delta_1^{-p_1} + K_1^{p_1} \right) K_1 \gamma^\alpha \leq \frac{1}{16},$$

as for instance,

$$\gamma := \left(\frac{1}{48C} \min \left\{ K_1^{-1}, K_1^{-1} \delta_1^{p_1}, K_1^{-(1+p_1)} \right\} \right)^{1/\alpha} \wedge 1. \quad (\text{B.28})$$

Then, we choose $\varepsilon > 0$ such that

$$C \left(1 + \delta_1^{-p_1} + K_1^{p_1} \right) \left(1 + \gamma^{-(2+\alpha)} \right) K_1 \varepsilon \leq \frac{1}{16}.$$

A suitable choice is

$$\varepsilon := \frac{1}{96C} (1 + \gamma^{2+\alpha}) \min \left\{ K_1^{-1}, K_1^{-1} \delta_1^{p_1}, K_1^{-(1+p_1)} \right\} \quad (\text{B.29})$$

Then, we obtain

$$\begin{aligned} [u_t]_{C^\alpha(\bar{Q})} &\leq \frac{1}{4} [u]_{C^{2+\alpha}(\bar{Q})} + C e^{\lambda_1 T} \left(1 + \delta_1^{-p_1} + K_1^{p_1} \right) (1 + K_1 \varepsilon^{-m}) \left(1 + \gamma^{-(2+\alpha)} \right) \\ &\quad \times \left(\|f\|_{C^\alpha(\bar{Q})} + \|g\|_{C^{2+\alpha}(\mathbb{R}^d)} \right). \end{aligned} \quad (\text{B.30})$$

By combining inequalities (B.17) and (B.30) of the preceding two cases, we obtain the global estimate

$$\begin{aligned} [u_t]_{C^\alpha(\bar{Q})} &\leq \frac{1}{4} [u]_{C^{2+\alpha}(\bar{Q})} + C e^{\lambda_1 T} \left(1 + \delta_1^{-p_1} + K_1^{p_1} \right) (1 + K_1 \varepsilon^{-m}) \left(1 + \gamma^{-(2+\alpha)} \right) \\ &\quad \times \left(\|f\|_{C^\alpha(\bar{Q})} + \|g\|_{C^{2+\alpha}(\mathbb{R}^d)} \right). \end{aligned} \quad (\text{B.31})$$

We notice from (B.28) and (B.29) that we may find positive constants $N_3 = N_3(d, \alpha, T)$ and $p = p(\alpha)$ such that

$$[u_t]_{C^\alpha(\bar{Q})} \leq \frac{1}{4} [u]_{C^{2+\alpha}(\bar{Q})} + N_3 e^{\lambda_1 T} \left(1 + \delta_1^{-p} + K_1^p \right) \left(\|f\|_{C^\alpha(\bar{Q})} + \|g\|_{C^{2+\alpha}(\mathbb{R}^d)} \right).$$

The similar argument applied to $[u_{x_i x_j}]_{C^\alpha(\bar{Q})}$ yields

$$[u_{x_i x_j}]_{C^\alpha(\bar{Q})} \leq \frac{1}{4} [u]_{C^{2+\alpha}(\bar{Q})} + N_3 e^{\lambda_1 T} \left(1 + \delta_1^{-p} + K_1^p \right) \left(\|f\|_{C^\alpha(\bar{Q})} + \|g\|_{C^{2+\alpha}(\mathbb{R}^d)} \right).$$

Therefore, (B.13) gives us

$$[u]_{C^{2+\alpha}(\bar{Q})} \leq N_3 e^{\lambda_1 T} \left(1 + \delta_1^{-p} + K_1^p \right) \left(\|f\|_{C^\alpha(\bar{Q})} + \|g\|_{C^{2+\alpha}(\mathbb{R}^d)} \right),$$

which concludes the proof of the proposition by the interpolation inequalities [24, Theorem 8.8.1] and the maximum principle estimate (B.15). \square

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